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Today will be a short lecture, because Bengt wants to go to London. There is some conference in string theory.

8. World sheet currents

§ 8.1: Electric charge conservation

Consider Maxwell's equations:

$$\partial_\nu F^{\mu\nu} = \frac{1}{c}\, j^\mu$$

Then, I hit this with ∂_μ:

$$\partial_\mu \partial_\nu F^{\mu\nu} = \frac{1}{c}\, \partial_\mu\, j^\mu$$

The left hand side is identically zero, since the indices on the derivatives commute, and we have $F^{\mu\nu} = -F^{\mu\nu}$.

$$\Rightarrow \quad \partial_\mu j^\mu = 0$$

So, we say that the current j^μ is conserved — although it is really the corresponding charge that is conserved. We define the charge as the space integral of the zero-component of the current:

$$Q := \frac{1}{c} \int_V \mathrm{d}^3x\, j^0$$

$j^0 \equiv c\ \rho$, where ρ is the ordinary electric charge density. Q, as defined above, is the charge enclosed in the volume V.

$$\dot{Q} = \frac{\mathrm{d}}{\mathrm{d}t} Q = \frac{1}{c} \int_V \mathrm{d}^3x\, \partial_0 j^0 = -\frac{1}{c} \int_V \mathrm{d}^3x \nabla \cdot \boldsymbol{j} = -\frac{1}{c} \int_{\partial V} \mathrm{d}\boldsymbol{\sigma} \cdot \boldsymbol{j}$$

Two situations:

1) Evaluate this at infinity, assuming that all matter distributions are contained in a finite volume: $\dot{Q} = 0$, because then $\boldsymbol{j} = \mathbf{0}$.

2) Consider a surface at finite distance:

$$\dot{Q} + \frac{1}{c} \int_{\partial V} \mathrm{d}\boldsymbol{\sigma} \cdot \boldsymbol{j} = 0$$

How do we get currents that behave (are conserved) like this?

§ 8.2: Lagrangian symmetries and Noether's theorem

Consider particle mechanics:

$$S = \int \mathrm{d}t\, L(q, \dot{q}; t)$$

where the t dependence of L comes from external "forces". Take the variation

$$\begin{cases} q \rightarrow q + \delta q \\ \dot{q} \rightarrow \dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\, \delta q \end{cases}$$

of the action S:

$$\delta S = \int \mathrm{d}t \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q} \right) = \int \mathrm{d}t \left(\frac{\partial L}{\partial q} - \partial_t \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q + \left[\frac{\partial L}{\partial \dot{q}}\,\delta q \right]_{t_1}^{t_2}$$

Now we have two options to use this result:

1. If δq is *any* variations (vanishing at t_1 and t_2)

$$\Rightarrow \begin{cases} \text{Lagrange equations of motion (bulk term)} \\ \text{Boundary terms } = 0, \delta q|_{t_1}^{t_2} \equiv 0. \end{cases}$$

2. Pick special δq that correspond to symmetries (these will *not* vanish at t_1 and t_2; often they are constant).

If $\delta_\varepsilon q$ is a symmetry $\delta_\varepsilon S = 0$. $\Rightarrow$ When the Lagrange equations are satisfied (sometimes we call this "being on-shell") then $\delta_\varepsilon S = 0 \Rightarrow$

$$\frac{\partial L}{\partial \dot{q}}\,\delta_\varepsilon q \text{ is } t \text{ independent.}$$

So define $\delta_\varepsilon q \equiv \varepsilon \Delta q$, $\varepsilon \equiv$ constant and

$$\varepsilon\, Q \equiv \frac{\partial L}{\partial \dot{q}} (\varepsilon\, \Delta q)$$

Then $\dot{Q} = 0$, where $Q \equiv \frac{\partial L}{\partial \dot{q}}\,\Delta q$.

Example: If $L = \frac{1}{2}\, m\, \dot{x}^2$ and

$$\delta_\varepsilon x = \varepsilon \Rightarrow \delta_\varepsilon \dot{x} = 0 \Rightarrow Q = \frac{\partial L}{\partial \dot{x}} = m\, \dot{x} = p$$

So space translation invariance $\Rightarrow$ momentum conservation!

Field theory

Example: Charged scalar field

$$\mathcal{L} = -\,\partial_\mu \bar{\phi}\, \partial^\mu \phi + m^2\, |\phi|^2 - V(|\phi|)$$

Symmetry: $\phi \rightarrow \mathrm{e}^{\mathrm{i}\alpha} \phi$ for a constant α. This will generate a current.

Notation:

Index α: ξ^α are coordinates on the worldsheet in string theory or in space time in field theory.

Index a, as in ϕ^a, labels the field components. $a =$ target space index in string theory.

Index i, as in ε^i, labels the symmetries.

$$\delta_\varepsilon S = 0 = \int \mathrm{d}^d \xi \left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) \right) \delta_\varepsilon \phi^a + \int \mathrm{d}^d \xi\, \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\, h_i^a(\phi) \right)$$

where we have defined $\delta_\varepsilon \phi^a = \varepsilon^i h_i^a(\phi)$

§ 8.3: Current on the string world sheet

$$S = -\frac{T_0}{c}\int \underbrace{\mathrm{d}\tau\,\mathrm{d}\sigma}_{\mathrm{d}^2\xi}\sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2(x')^2}$$

There is an obvious symmetry, namely $\delta_\varepsilon x^\mu = \varepsilon^\mu = \text{constant}$. The corresponding currents are

$$P^\alpha{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha x^\mu)}$$

The conservation equation says

$$\partial_\alpha P^\alpha{}_\mu = 0.$$

But this is just the field equation. This was the equation of motion that we derived before. Also the "charge" p_μ is

$$p_\mu = \int_0^{\sigma_1} \mathrm{d}\sigma P^\tau{}_\mu$$

satisfies $\dot{p}_\mu = 0$ if we have Neumann boundary conditions.

§ 8.4: The complete momentum current

$$p_\mu \equiv \int_0^{\sigma_1} \mathrm{d}\sigma P^\tau{}_\mu$$

is computed for a fixed τ, but this is not necessary:

$$p_\mu \equiv \int_\gamma (P^\tau{}_\mu\,\mathrm{d}\sigma - P^\sigma{}_\mu\,\mathrm{d}\tau)$$

where γ is any non-trivial path across the open string.

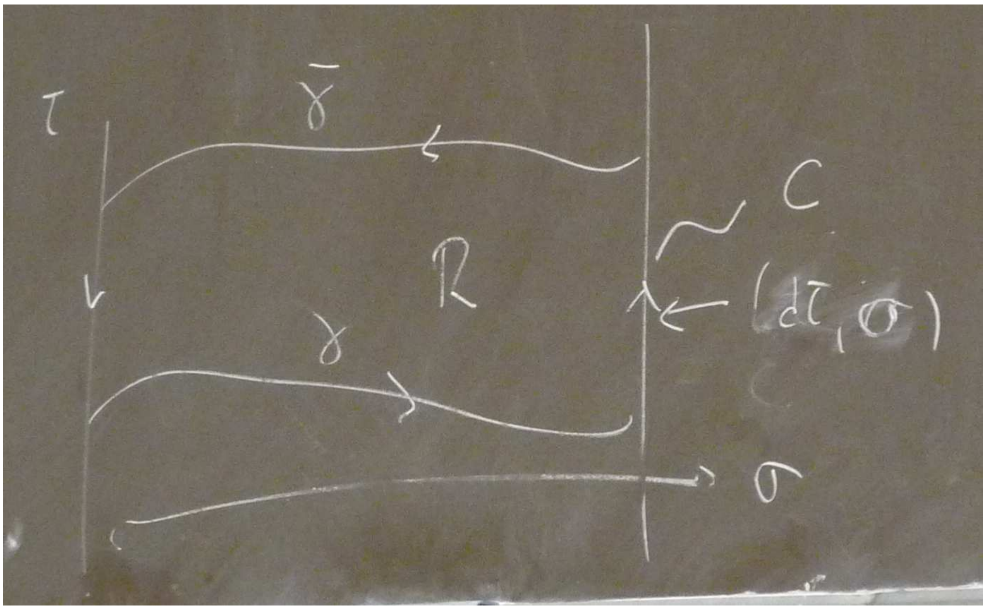

Figure 1.

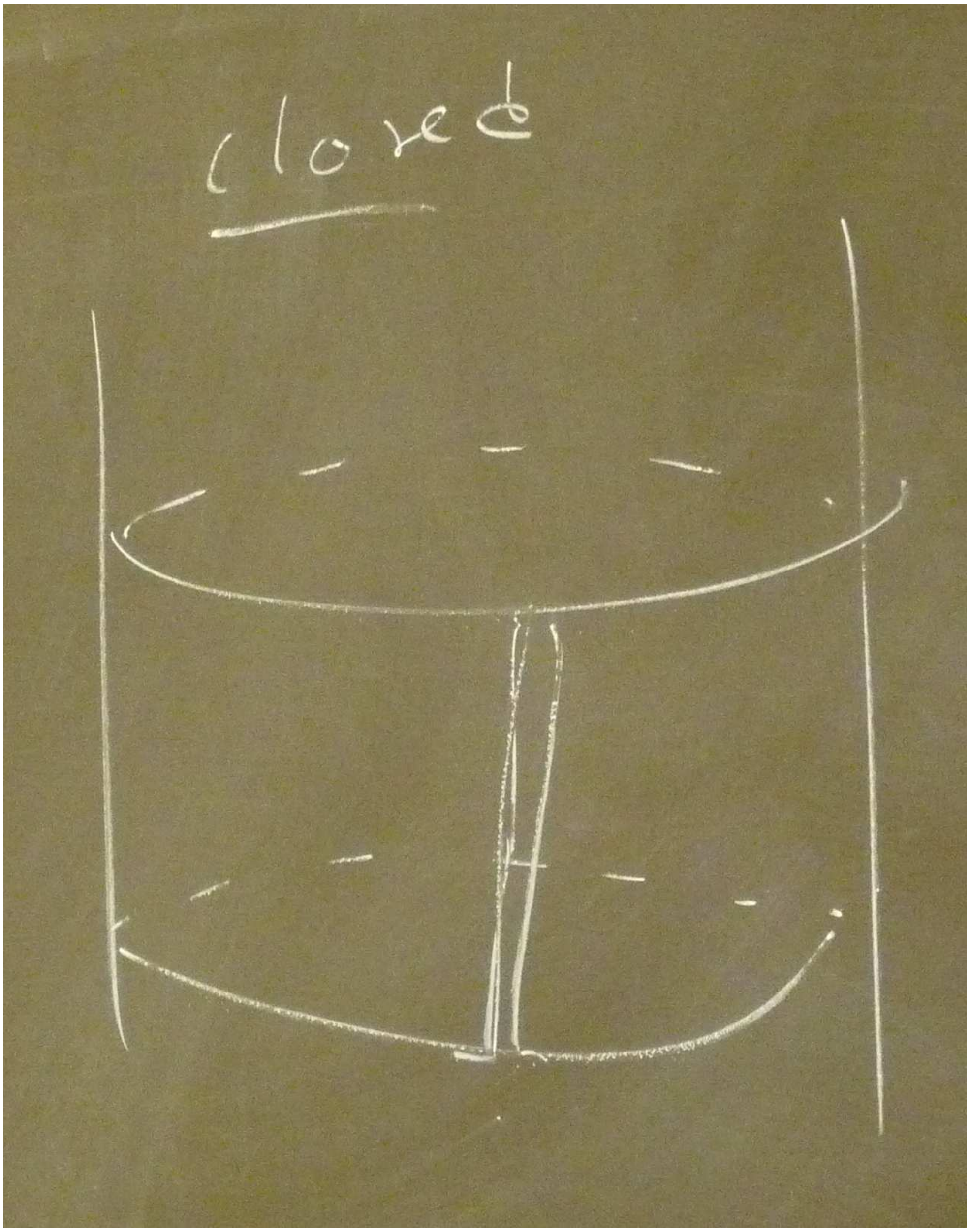

Figure 2.

First

$$\oint_{C=\partial R} (P^{\tau}{}_{\mu}\,\mathrm{d}\sigma - P^{\sigma}{}_{\mu}\,\mathrm{d}\tau) = \int_R (\partial_\tau P^{\tau}{}_{\mu} + \partial_\sigma P^{\sigma}{}_{\mu})\mathrm{d}\tau\mathrm{d}\sigma$$

and second note that the edges contribute nothing for Neumann boundary conditions.

$$\int_\gamma = \int_{\bar{\gamma}}$$

§ 8.6: α': The Regge slope parameter.

Read the rest yourself.