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6. Relativistic strings

Recall: The relativistic point particle

$$S = -\,m\,c \int_P \mathrm{d}s + \frac{q}{c}\int_P A - \frac{1}{4\,c}\int \mathrm{d}^D x\, F_{\mu\nu} F^{\mu\nu}$$

where P is the world line $x^\mu(\tau)$ and $A = A_\mu(x(\tau))\,\mathrm{d}x^\mu = A_\mu(x(\tau))\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau$. Without the last term the particle moves in a fixed background field.

If you vary with respect to $x^\mu(\tau)$ you get the equations of motion for the particle. If you vary $A_\mu(x)$ — no τ — you get Maxwell's equations with source terms. (δ-functions for point charges, since the source term is the density.)

The first term we want to understand in the string context is the first one, $-\,m\,c \int_P \mathrm{d}s$. We will get to the others later.

§ 6.1: Spatial soap films

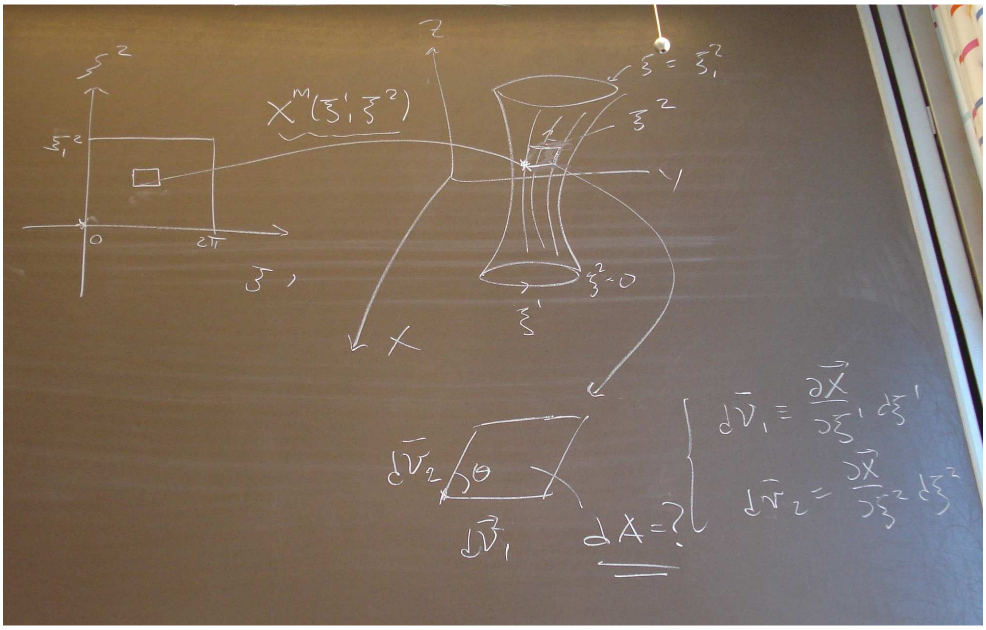

Figure 1. We have a parameter space with ξ^i and a target space with coordinates x^m. A square on the parameter space maps to a square on the soap film that has to match the coordinate lines. The map is $x^m(\xi^1, \xi^2)$. We want to compute the area element $\mathrm{d}A$.

$$\begin{cases} \mathrm{d}\boldsymbol{v}_1 \equiv \frac{\partial \boldsymbol{x}}{\partial \xi^1}\,\mathrm{d}\xi^1 \\ \mathrm{d}\boldsymbol{v}_2 \equiv \frac{\partial \boldsymbol{x}}{\partial \xi^2}\,\mathrm{d}\xi^2 \end{cases}$$

$$\mathrm{d}A = |\mathrm{d}\boldsymbol{v}_1| \cdot |\mathrm{d}\boldsymbol{v}_2| \cdot \sin\theta = |\mathrm{d}\boldsymbol{v}_1| \cdot |\mathrm{d}\boldsymbol{v}_2| \cdot \sqrt{1 - \cos^2\theta} = \left(|\mathrm{d}\boldsymbol{v}_1|^2 + |\mathrm{d}\boldsymbol{v}_2|^2 - |\mathrm{d}\boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{v}_2|^2\right)^{1/2}$$

Introduce the notation: $\xi^i = \left(\xi^1, \xi^2\right)$:

$$g_{ij} := \frac{\partial x^\mu}{\partial \xi^i}\,\frac{\partial x^n}{\partial \xi^j}\,\delta_{mn}$$

Then $\mathrm{d}A = \sqrt{\det(g_{ij})}\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2$. g_{ij} here is called the "induced metric" or the "pull-back". g_{ij} is the induced metric from the target space (with metric δ_{mn}) via the embedding functions $x^m(\xi)$.

So the action here is

$$S = \int \mathrm{d}A = \int \sqrt{\det(g_{ij})}\, \mathrm{d}^2\xi$$

This is the invariant measure (or volume) from general relativity. Let me quickly prove this to you.

Invariance under general coordinate transformations on the surface: $\xi^i \to \tilde{\xi}^i = \tilde{\xi}^i(\xi)$. Now I want to show that the integral does not depend on the choice of coordinates — there is no physics in the choice of coordinates.

$$\Rightarrow \quad \frac{\partial x^m}{\partial \xi^i} = \frac{\partial x^m}{\partial \tilde{\xi}^j} \frac{\partial \tilde{\xi}^j}{\partial \xi^i} \quad \Rightarrow \quad g_{ij} = \frac{\partial \tilde{\xi}^k}{\partial \xi^i} \frac{\partial \tilde{\xi}^l}{\partial \xi^j} \underbrace{\left(\frac{\partial x^m}{\partial \tilde{\xi}^k} \frac{\partial x^n}{\partial \tilde{\xi}^l} \delta_{mn} \right)}_{\equiv \tilde{g}_{kl}}$$

Note the indices: $i, j = 1, 2$ and $m, n, \ldots = 1, \ldots, d$. So

$$g_{ij} = \frac{\partial \tilde{\xi}^k}{\partial \xi^i} \frac{\partial \tilde{\xi}^l}{\partial \xi^j} \tilde{g}_{kl}$$

So, now we take the determinant of this:

$$\det(g) = \left| \det\left(\frac{\partial \tilde{\xi}}{\partial \xi} \right) \right|^2 \det(\tilde{g})$$

$$\Rightarrow \sqrt{\det(g)} = \left| \det\left(\frac{\partial \tilde{\xi}}{\partial \xi} \right) \right| \sqrt{\det(\tilde{g})}$$

Use then the fact that:

$$\mathrm{d}\xi^1\, \mathrm{d}\xi^2 = \left| \det\left(\frac{\partial \xi^i}{\partial \tilde{\xi}^j} \right) \right| \mathrm{d}\tilde{\xi}^1\, \mathrm{d}\tilde{\xi}^2$$

(This is best done using wedge products, an anti-symmetric product. $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} = \varepsilon^{\mu_1 \ldots \mu_D}\, \mathrm{d}^D x$.

$$\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 = \left(\frac{\partial \xi^1}{\partial \tilde{\xi}^1} \mathrm{d}\tilde{\xi}^1 + \frac{\partial \xi^1}{\partial \tilde{\xi}^2} \mathrm{d}\xi^2 \right) \wedge \left(\frac{\partial \xi^2}{\partial \tilde{\xi}^1} \mathrm{d}\tilde{\xi}^1 + \frac{\partial \xi^2}{\partial \tilde{\xi}^2} \mathrm{d}\tilde{\xi}^2 \right) =$$

$$= \det\left(\frac{\partial \xi}{\mathrm{d}\tilde{\xi}} \right) \mathrm{d}\tilde{\xi}^1 \wedge \mathrm{d}\tilde{\xi}^2$$

We have $|\mathrm{d}\tilde{\xi}^1 \wedge \mathrm{d}\tilde{\xi}^2| \equiv \mathrm{d}^2\xi$. QED)

But then

$$\mathrm{d}^2\xi \sqrt{\det(g)} = \mathrm{d}^2\tilde{\xi} \sqrt{\det(\tilde{g})}$$

$$\left(\text{Use } \frac{\partial \tilde{\xi}^i}{\partial \xi^j} \frac{\partial \xi^j}{\partial \tilde{\xi}} = \delta^i{}_k\right)$$

$$\det\left(\frac{\partial \tilde{\xi}}{\partial \xi} \right) \det\left(\frac{\partial \xi}{\partial \tilde{\xi}} \right) = 1$$

§ 6.3: Area functional for space-time surfaces. (Space surfaces are just strings: string theory.)

I need to introduce some words. We have a map $X^\mu(\tau,\sigma)$. τ and σ are now the coordinates on the world sheet. This X is called the *string coordinate*.

$$X^\mu(\tau,\sigma):\quad \Sigma_2 \to M_D$$

Σ_2 is two-dimensional. M_D is the target space time. It can be Minkowski, de Sitter, anti-de Sitter, Minkowski with black holes, or whatever! M_D is described with lower case $(x^0, x^1, ..., x^d)$.

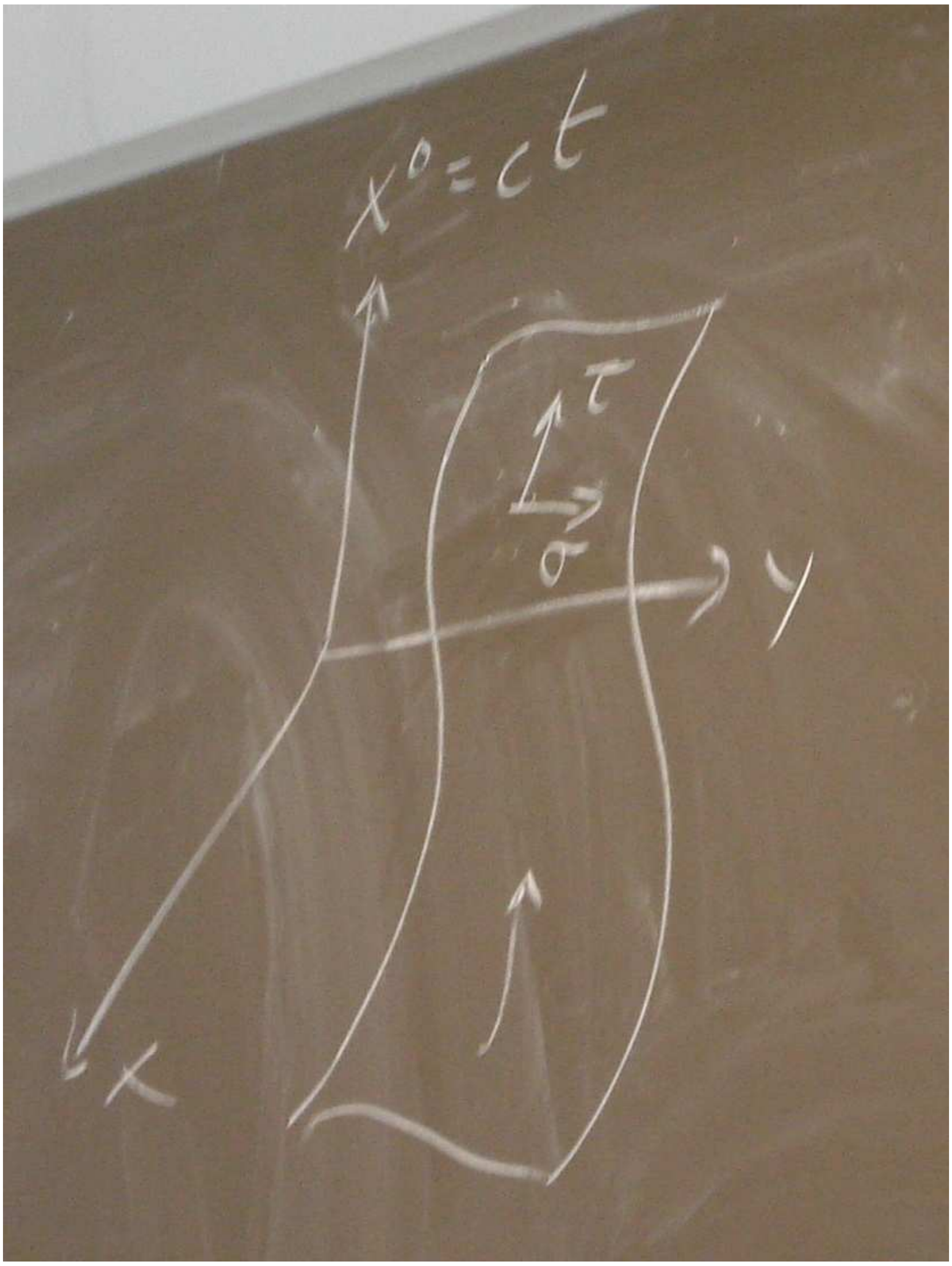

Figure 2. The string traces out a surface when it moves in space-time.

We require $\frac{\partial x^0}{\partial \tau} > 0$.

$$A = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{-\det(\gamma_{\alpha\beta})}$$

where $\gamma_{\alpha\beta}$ is given by, with $\xi^\alpha = (\tau, \sigma)$:

$$\gamma_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \eta_{\mu\nu}$$

The minus sign of $\sqrt{-\det(\gamma_{\alpha\beta})}$ is because the determinant is negative in relativity.

Note that any small but finite section of the string has non-zero mass ($T_0\, \mathrm{d}s$) which means that it moves with velocity $v < c$! The end points are special, and if there is a kink, that special point could also move with the speed of light.

§ 6.4: The Nambu–Goto action

We use units

$$\begin{cases} [\tau] = [t] = T \\ [\sigma] = [x^\mu] = L \\ [A] = L^2 \end{cases}$$

$$\Rightarrow S = -\frac{T_0}{c} \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \sqrt{-\det(\gamma_{\alpha\beta})}$$

Then using

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}, \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma}$$

$$\Rightarrow \quad S = -\frac{T_0}{c} \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2\, (x')^2}$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{x}^\mu \dot{x}_\mu & \dot{x}^\mu x'_\mu \\ \dot{x}^\mu x'_\mu & x'^\mu x'_\mu \end{pmatrix}$$

§ 6.5: Equations of motion

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \mathcal{L}(\dot{x}, x')$$

Vary with respect to x^μ:

$$\delta x^\mu \quad \Rightarrow \quad \delta S = \int \mathrm{d}\tau\, \mathrm{d}\sigma \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta x'^\mu \right) =$$

$$= -\int \mathrm{d}\tau\, \mathrm{d}\sigma \left(\partial_\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) + \partial_\sigma \frac{\partial \mathcal{L}}{\partial x'^\mu} \right) \delta x^\mu + \int \mathrm{d}\tau\, \mathrm{d}\sigma \left(\underbrace{\partial_\tau \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta x^\mu \right)}_{=0;\, \delta x^\mu|_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} = 0} + \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial x'^\mu} \delta x^\mu \right) \right)$$

$\delta S = 0$: Bulk term:

$$\partial_\tau P^\tau{}_\mu + \partial_\sigma P^\sigma{}_\mu = 0, \quad \text{where } P^\tau{}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \text{ and } P^\sigma{}_\mu = \frac{\partial \mathcal{L}}{\partial x'^\mu}.$$

Boundary term: At $\sigma = 0$ and $\sigma = \sigma_1$ we have $P^\sigma{}_\mu \cdot \delta x^\mu = 0$. How to satisfy this condition? We have to look at each end and each coordinate x^μ separately:

1. Space: x^m. Then we can have either $\delta x^m = 0$, and that's of course Dirichlet. Or we can have $P^\sigma{}_m = 0$. This is not Neumann. Free end condition. (It becomes Neumann after gauge fixing!)

2. Time: x^0. Then we can only have $P^\sigma{}_0 = 0$: Dirichlet is impossible, because $\delta x^0 \neq 0$. You can't fix time.

Note: The $P^\sigma{}_\mu$ are very complicated $\rightarrow$ need to be simplified!

D-branes

Example: Recall the picture

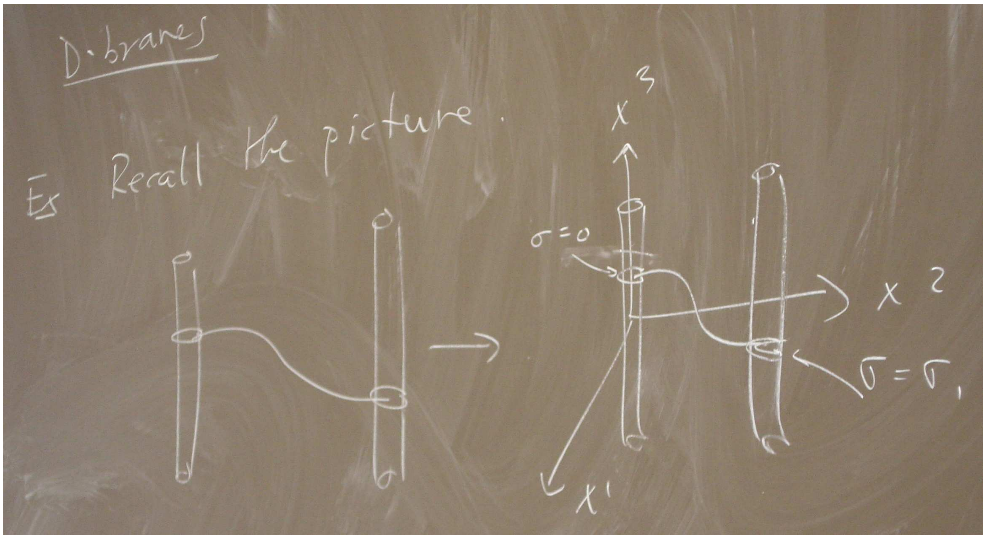

Figure 3.

Boundary conditions: We have Dirichlet boundary conditions at both ends of x^1 and x^2 but

Neumann (free end) at both ends of x^3. We call this object a D1-brane. This object has a size.

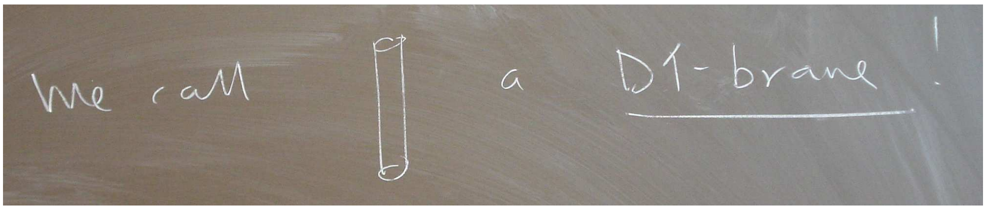

Figure 4.

D-branes are not point-like, they have size coming from a horizon or something like that.

Another example:

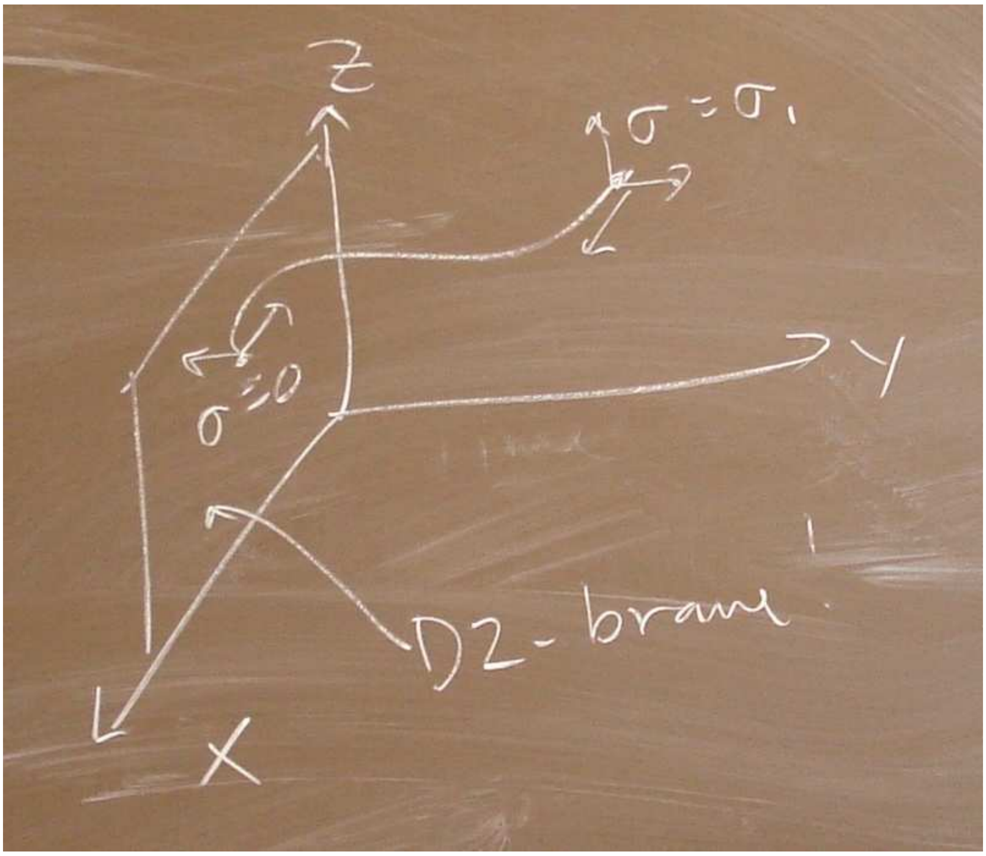

Figure 5. The endpoint of the string is free to move on the D2-brane.

$\sigma =$	0	σ_1
x	Neumann	Neumann
y	Dirichlet	Neumann
z	Neumann	Neumann
(fig. 5)	D2-brane	D3-brane (space filling)

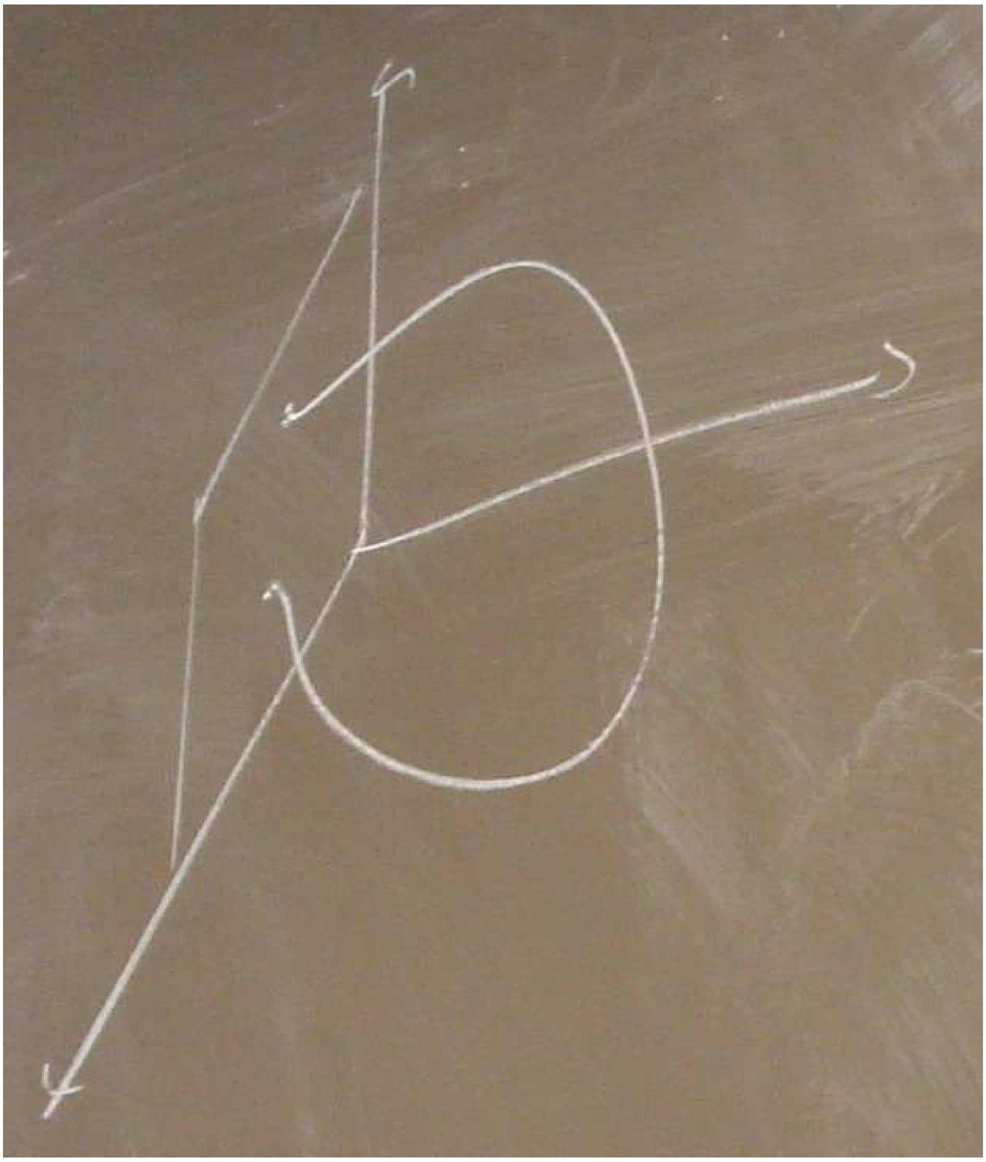

Figure 6.

$\sigma =$	0	σ_1
x	Neumann	Neumann
y	Dirichlet	Dirichlet
z	Neumann	Neumann

(cf. fig. 6)

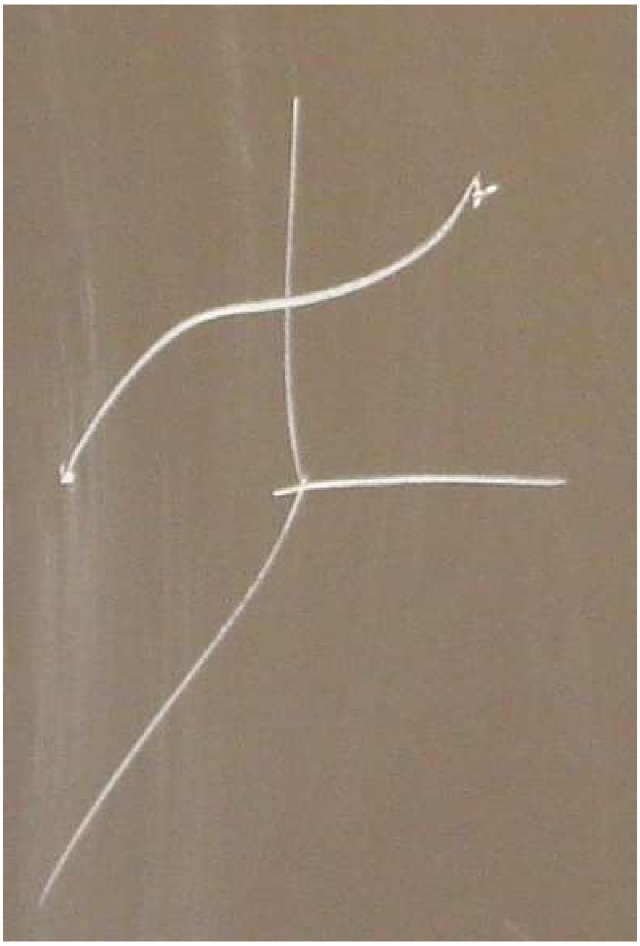

Figure 7. Space-filing branes at both ends: *open strings!*

D-branes

- Physical objects (soliton, black holes — black holes can be considered solitons in some sense.)
- Can be infinite, and infinitely heavy.
- Can be finite sized.
- Infinitely extended D-branes may look like point particles in $D = 4$ spacetime after compactification.

§ 6.6: The static gauge

Since the world sheet coordinates $\xi^\alpha = (\tau, \sigma)$ are arbitrary we might pick a special set to get simpler expressions for $(P^\tau{}_\mu, P^\sigma{}_\mu)$. Consider a particular Lorentz frame and for any point Q on the world surface we set $\tau(Q) = t(Q)$.

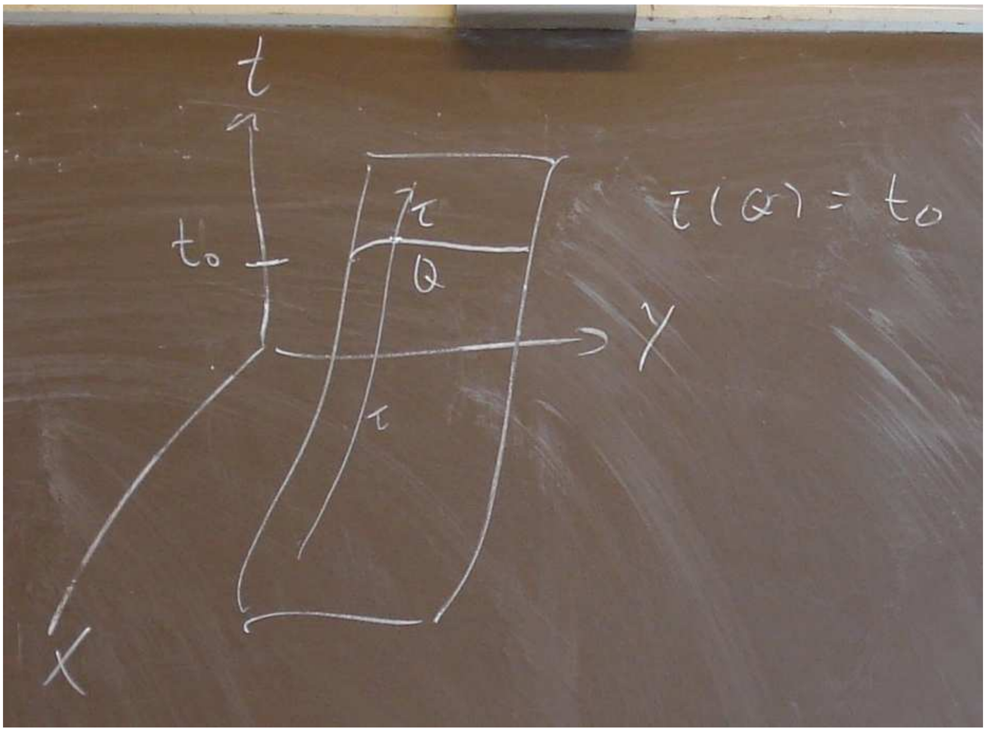

Figure 8.

This is called a static gauge: $x^0(\tau,\sigma) = c\,t(\tau,\sigma) = c\,\tau$ so $t(\tau,\sigma) = \tau$.

$$\Rightarrow x^\mu(\tau,\sigma) = (c\,t, \boldsymbol{x}(\tau,\sigma))$$

$$\Rightarrow \frac{\partial x^\mu}{\partial \tau} = \left(c, \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{v}\right), \quad \frac{\partial x^\mu}{\partial \sigma} = \left(0, \frac{\partial \boldsymbol{x}}{\partial \sigma} = \boldsymbol{x}'\right)$$

There is a square root in the Nambu-Goto action. Check $\sqrt{>0}$. Set $\boldsymbol{v} = 0$.

$$(\dot{x} \cdot x')^2 - \dot{x}^2\,(x')^2 = 0 - (-c^2)(\boldsymbol{x}')^2 > 0: \quad \text{OK.}$$

§ 6.7: Tension and energy

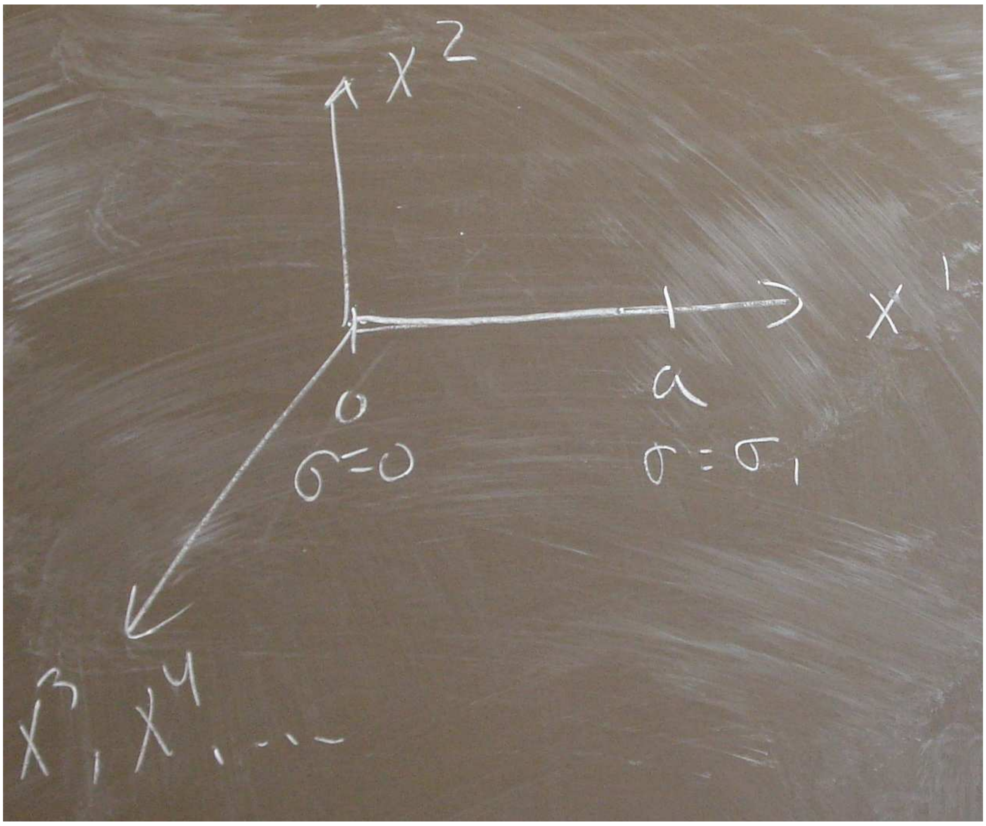

Figure 9.

Space vector: $\sigma = 0$: $(0, \mathbf{0})$ and $\sigma = \sigma_1$: $(a, \mathbf{0})$ where, in this context, $\mathbf{0} = (x^2, x^3, \ldots, x^d)$.

Next let's evaluate the action in the static gauge:

$$\begin{cases} x^0 = c\,t = c\,\tau \\ x'(\tau,\sigma) = f(\sigma) \\ x^2 = x^3 = \ldots = x^d = 0 \end{cases} \Rightarrow \frac{\partial x'}{\partial\sigma} = f'(\sigma)$$

$$\Rightarrow \quad S = -\frac{T_0}{c}\int \underbrace{\mathrm{d}\tau}_{=\mathrm{d}t}\,\mathrm{d}\sigma\sqrt{c^2(f'(\sigma))^2} = -T_0\int \mathrm{d}t\,\mathrm{d}\sigma\sqrt{f'(\sigma)^2} = -T_0\int_{t_\mathrm{i}}^{t_\mathrm{f}}\mathrm{d}t\int_0^{\sigma_1}\mathrm{d}\sigma\,\frac{\mathrm{d}}{\mathrm{d}\sigma}f(\sigma) =$$

$$= -T_0(t_\mathrm{f}-t_\mathrm{i})\Bigg(\underbrace{f(\sigma_1)}_{=a} - \underbrace{f(0)}_{=a}\Bigg) = -T_0\,a\,(t_\mathrm{f}-t_\mathrm{i})$$

$$= \int \mathrm{d}t\,L = \int \mathrm{d}t\Bigg(\underbrace{T}_{=0} - V\Bigg)$$

We get

$$V = T_0\,a \quad \text{and} \quad \mu_0 c^2 = \frac{V}{a} = T_0$$

Important check: Does the string we consider here satisfy the equations of motion. That's absolutely necessary, of course. Does it satisfy the proper boundary conditions? (We don't have to check here, for Dirichlet boundary conditions.) Check

$$\partial_\tau P^\tau{}_\mu + \partial_\sigma P^\sigma{}_\mu = 0: \quad \text{OK.}$$

§ 6.8: Action in terms of transverse velocity.

Static gauge: $t = \tau$.

$$x^\mu(\tau,\sigma) = (c\,t, \boldsymbol{x}(t,\sigma))$$

$$\Rightarrow \quad \begin{cases} \dot{x}^\mu = (c, \dot{\boldsymbol{x}}) \\ x'^\mu = (0, \boldsymbol{x}') \end{cases}$$

$$\dot{x}^2 = -c^2 + \dot{\boldsymbol{x}}^2, \quad (x')^2 = (\boldsymbol{x}')^2$$

$$\dot{x}\cdot x' = \dot{\boldsymbol{x}}\cdot\boldsymbol{x}'$$

$$\Rightarrow \left(\dot{x}\cdot x'\right)^2 - \dot{x}^2(x')^2 = (\dot{\boldsymbol{x}}\cdot\boldsymbol{x}')^2 + \left(c^2 - \dot{\boldsymbol{x}}^2\right)(\boldsymbol{x}')^2$$

Now fix t and let $\mathrm{d}s$ denote the spacelike length element, $\mathrm{d}s \equiv |\mathrm{d}\boldsymbol{x}(\sigma)| = \left|\frac{\partial\boldsymbol{x}}{\partial\sigma}\right|\,|\mathrm{d}\sigma|$. For instance

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = 1.$$

$$\Rightarrow \quad (\dot{x}\cdot x')^2 - \dot{x}^2(x')^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2\left(\left(\frac{\partial\boldsymbol{x}}{\partial t}\frac{\partial\boldsymbol{x}}{\partial s}\right)^2 + \left(c^2 - \left(\frac{\partial\boldsymbol{x}}{\partial t}\right)^2\right)\left(\frac{\partial\boldsymbol{x}}{\partial s}\right)^2\right) =$$

$$= \left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2\left(c^2 - \Bigg(\underbrace{\left(\frac{\partial\boldsymbol{x}}{\partial t}\right) - \left(\frac{\partial\boldsymbol{x}}{\partial t}\cdot\frac{\partial\boldsymbol{x}}{\partial s}\right)}_{=\boldsymbol{v}_\perp}\Bigg)^2\right) = \left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2(c^2 - \boldsymbol{v}_\perp^2)$$

$$\boldsymbol{v}_\perp = \frac{\partial\boldsymbol{x}}{\partial t} - \left(\frac{\partial\boldsymbol{x}}{\partial t}\cdot\frac{\partial\boldsymbol{s}}{\partial s}\right)\frac{\partial\boldsymbol{x}}{\partial s}$$

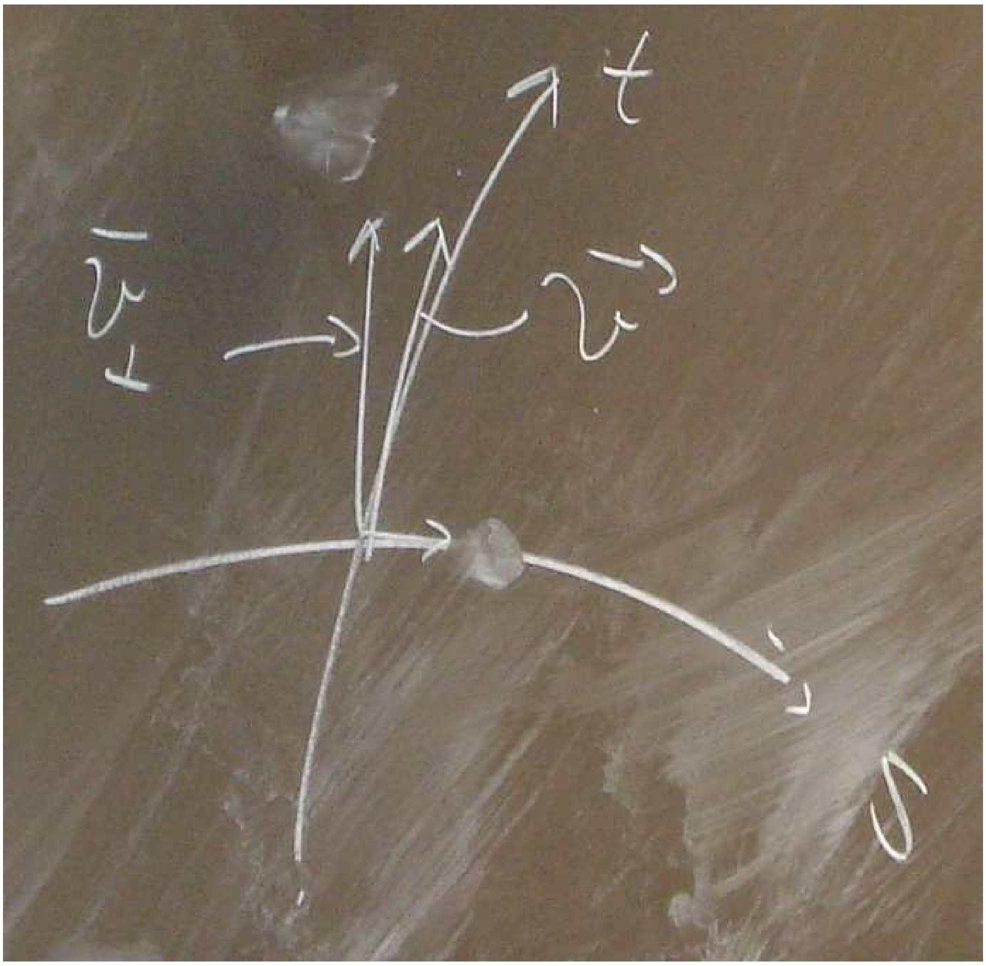

Figure 10.

$$S = -T_0 \int \mathrm{d}t \int_0^{\sigma_1} \mathrm{d}\sigma \, \frac{\mathrm{d}s}{\mathrm{d}\sigma} \sqrt{1 - \frac{\boldsymbol{v}_\perp^2}{c^2}}$$

This $\sqrt{1 - \boldsymbol{v}_\perp^2 / c^2}$ confirms the picture of the fundamental string!

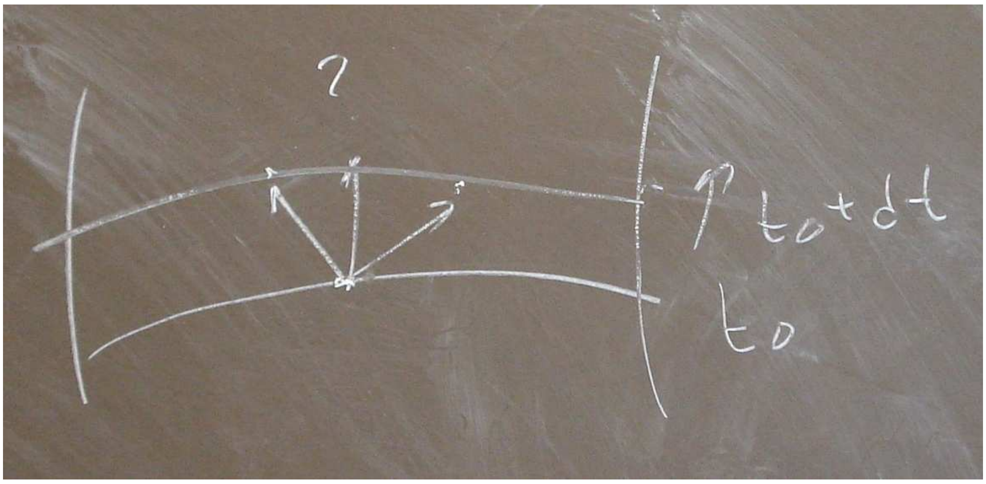

Figure 11.

§ 6.9: The endpoints of open strings

Boundary conditions: Neumann: ${P^\sigma}_\mu = 0$ at $\sigma = 0$ and $\sigma = \sigma_1$. If we plug the static gauge expressions

$$\begin{cases} \frac{\partial \boldsymbol{x}}{\partial s} \cdot \frac{\partial \boldsymbol{s}}{\partial t} = 0 \\ v = c \end{cases}$$

You should do this. This tells you that the ends move perpendicular to the string, with the velocity of light.