2009–03–24

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Chapter 4: Non-relativistic strings

Recall:

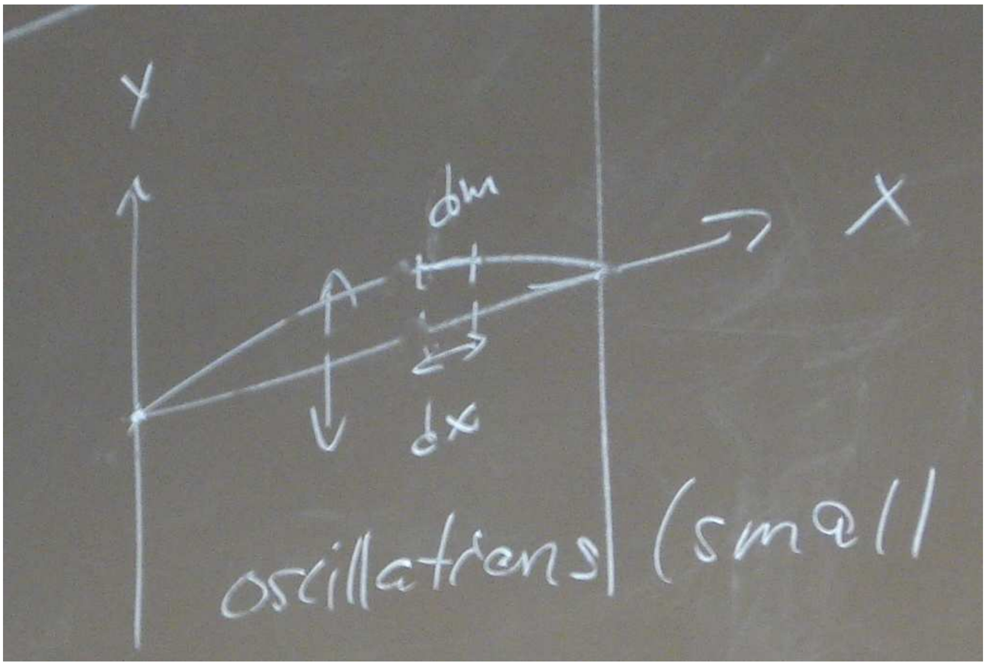

Figure 1. $\mathrm{d}m = \mu_0\,\mathrm{d}x$

Oscillations only in transverse directions (i.e. y-direction). Small: $\frac{\partial y}{\partial x} \ll 1$.

$$\Rightarrow L = T - V$$

where

$$\begin{cases} T = \displaystyle\int_0^a \frac{1}{2}(\mu_0\,\mathrm{d}x)\dot{y}^2 \\ V = \displaystyle\int_0^a T_0\,\mathrm{d}l \end{cases}$$

$\mathrm{d}l$ is the stretching, the additional length of the stretched string over the string in its ground state: $\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} - \mathrm{d}x \approx \frac{1}{2} \cdot \frac{\partial^2 y}{\partial x^2}\,\mathrm{d}x$.

$$\begin{cases} T = \displaystyle\int_0^a \frac{1}{2}\,\mu_0\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \\ V = \displaystyle\int_0^a \frac{1}{2}\,T_0\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \end{cases}$$

Chapter 5: The relativistic point particle

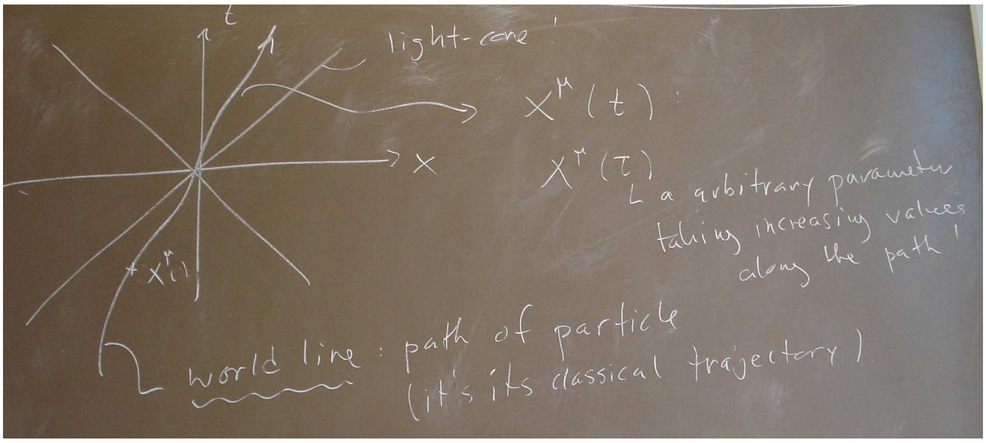

Figure 2. Light cone. World line: path of particle (it's its classical trajectory). Parametrise with coordinate time t as $x^\mu(t)$ or with an arbitrary parameter τ as $x^\mu(\tau)$. τ increases monotonously along the path.

Note: A free particle has a world line that is just a straight line.

Try

$$S = -m\,c\int_P \mathrm{d}s,$$

where $\mathrm{d}s = \sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}$ and m is the invariant mass (the rest mass).

Why is this correct?

- Dimension: $[S] = \text{action} = \mathrm{N\,m\,s}$ (energy integrated over time, compare $\int\ \mathrm{d}t\,\frac{1}{2}m\,v^2$.)
- Different Lorentz observers have to agree on the physics.
- It should reduce to the non-relativistic answer for $v \ll c$.

$$S_{\text{non-rel}} = \int \mathrm{d}t\,\frac{1}{2}m\,v^2.$$

$$S = -m\,c\int\sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} = -m\,c\int\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,\mathrm{d}t$$

$$\Rightarrow S = -m\,c^2\int\sqrt{1-\frac{v^2}{c^2}}\,\mathrm{d}t \overset{v\ll c}{\simeq} -m\,c^2\int\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)\mathrm{d}t =$$

$$= \underbrace{-m\,c^2(t_{\mathrm{f}}-t_{\mathrm{i}})}_{?} + \underbrace{\int\frac{1}{2}m\,v^2\,\mathrm{d}t}_{\text{correct}}$$

The first term is not a problem, since it does not contribute to the equations of motion.

- Canonical momenta:

$$\boldsymbol{p} \equiv \frac{\partial L}{\partial\dot{\boldsymbol{x}}} = \frac{\partial L}{\partial\boldsymbol{v}} = \frac{m\,\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}:\quad \text{OK!}$$

- Hamiltonian:

$$H \equiv \boldsymbol{p}\cdot\boldsymbol{x} - L = \frac{m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}}:\quad \text{OK!}$$

§ 5.2: Reparametrisation invariance

$$S = -m\,c\int\mathrm{d}s = -m\,c\int\frac{\mathrm{d}s}{\mathrm{d}\tau}\,\mathrm{d}\tau = -m\,c\int\frac{\mathrm{d}s}{\mathrm{d}\tau'}\,\mathrm{d}\tau'$$

The number S will not change under a change of coordinates. Here $\tau' = \tau'(\tau)$ satisfying

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau} > 0.$$

Check:

$$S = -\,m\,c \int_{\tau_1}^{\tau_2} \sqrt{-\,\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau'}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau'}}\,\mathrm{d}\tau' = \left[\mathrm{d}\tau' = \left(\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}\right)\mathrm{d}\tau\right] = -\,m\,c \int_{\tau_1}^{\tau_2} \sqrt{-\,\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,\mathrm{d}\tau.$$

We say that $S = -\,m\,c\int\ \mathrm{d}s$ is *manifestly* reparametrisation invariant.

§ 5.3: Equations of motion

The variational principle:

$$\delta x^\mu \quad \Rightarrow \quad \delta S = -\,m\,c\int \delta(\mathrm{d}s) = -\,m\,c\int \delta\Big(\sqrt{-\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\Big) =$$

$$-\,m\,c\int\ \frac{1}{2}\frac{(-\,2\eta_{\mu\nu}\,\mathrm{d}x^\mu\delta(\mathrm{d}x^\nu))}{\sqrt{-\,\eta_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu}} = \left[\mathrm{d}x^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau\right] =$$

$$= m\,c\int\ \frac{\eta_{\mu\nu}\,\dot{x}^\mu\,\delta(\dot{x}^\nu)}{\sqrt{-\,\dot{x}^\mu\dot{x}_\mu}}\,\mathrm{d}\tau = \left[\begin{array}{c}\delta(\dot{x}^\nu) = \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu)\\ \text{integrate by parts}\end{array}\right] =$$

Before that: $\sqrt{-\,\dot{x}^\mu\dot{x}_\mu} = \sqrt{\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2} = \frac{\mathrm{d}s}{\mathrm{d}\tau} > 0$, and also $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\tau} = u^\mu\frac{\mathrm{d}s}{\mathrm{d}\tau}$

$$\Rightarrow \quad \delta S = m\,c\int \eta_{\mu\nu}\,u\,\delta\dot{x}^\nu\,\mathrm{d}\tau = c\int \eta_{\mu\nu}\,p^\mu\,\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{d}x^\nu)\,\mathrm{d}\tau = [\text{integrate by parts}] =$$

$$= -\,c\int \eta_{\mu\nu}\,\dot{p}^\mu\,\delta x^\nu\,\mathrm{d}\tau + \underbrace{c[\eta_{\mu\nu}p^\mu\,\delta x^\nu]_{\tau_1}^{\tau_2}}_{\equiv 0}$$

(The variation δx^μ vanishes at the endpoints of the time interval.)

$\delta S = 0$, the bulk term:

$$\dot{p}^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0: \quad \text{OK!}$$

Note:

$$\Rightarrow \quad \frac{\mathrm{d}p^\mu}{\mathrm{d}s} = 0 \quad \overset{p^\mu = m u^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}s}}{\Rightarrow} \quad \boxed{\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} = 0}$$

Note:

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \quad \left\{\begin{array}{ll}\text{4-vector in Minkowski space:} & x^\mu \to L^\mu{}_\nu\,x^\nu\\ \text{scalar on the world line:} & \text{changes of coordinates on the world line}, \tau.\end{array}\right.$$

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \quad \left\{\begin{array}{l}\text{4-vector}\\ \text{``vectors'' on world-line (compare } \partial_\mu)\end{array}\right.$$

$$\partial_\tau = \frac{\partial\tau'}{\partial\tau}\,\partial_{\tau'}$$

also

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = (c, \boldsymbol{v}) \quad \{\ \text{not a 4-vector}$$

§ 5.4: Relativistic particle with charge

Recall: Lorentz force law:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)$$

In a relativistic formulation we need $\boldsymbol{E}, \boldsymbol{B} \to F_{\mu\nu}, \boldsymbol{u} \to u^\mu$:

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}s} = \frac{q}{c} F_{\mu\nu} u^\nu$$

The corresponding action:

$$S = -m\,c \int_P \mathrm{d}s + \frac{q}{c} \int_P A_\mu u^\mu \mathrm{d}s$$

$\delta x^\mu \Rightarrow \delta S = 0$: Lorentz force law! $A_\mu = A_\mu(x(\tau))$. You have to vary the argument of A_μ, and that yields a derivative. *Please do it!*

$F_{\mu\nu}$ is a background (fixed field). If we want dynamics for the electromagnetic field (Maxwell's equations) we need to add the Maxwell term:

$$S = m\,c \int_P \mathrm{d}s + \frac{q}{c} \int_P A - \frac{1}{4c} \int_{\text{Mink.}} F_{\mu\nu} F^{\mu\nu}\, \mathrm{d}^D x$$

where $A = A_\mu \mathrm{d}x^\mu$ is a 1-form (differential geometry). Of course, $F_{\mu\nu}$ depends on x, but *not* on $x(\tau)$.

$$\begin{cases} \delta x^\mu \Rightarrow \text{Lorentz law} \\ \delta A \Rightarrow \text{Maxwell's equations} \end{cases}$$

To vary the $\int A$ part with δA, we need to make the integration go over all of Minkowski space.

$$\int A \to \int_P \int \mathrm{d}^4 x\, \delta(x - x(\tau))\, A(x)$$

Chapter 6: Relativistic strings

§ 6.1: Area functionals

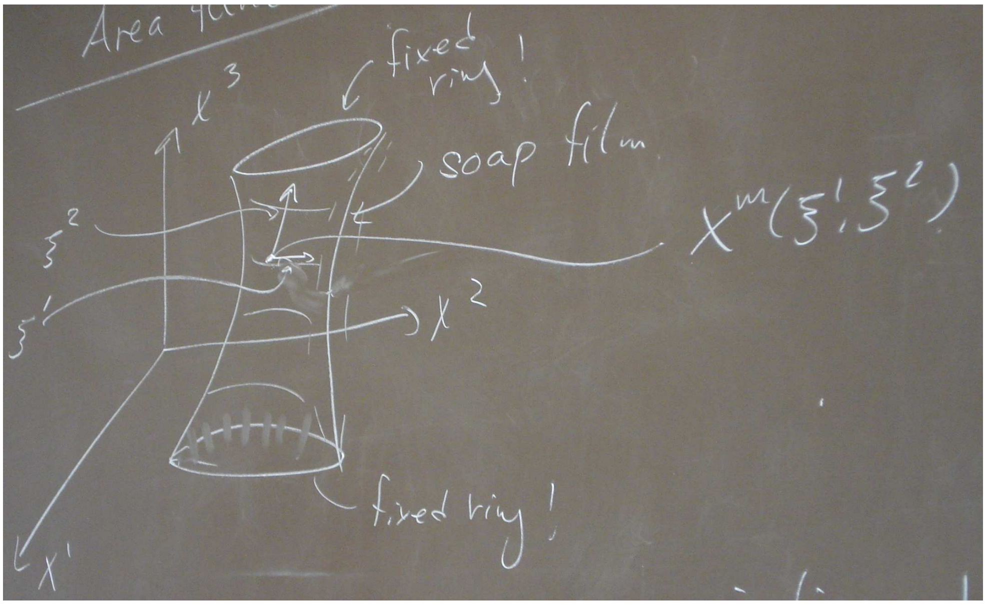

Figure 3. A soap film between two fixed rings.

The shape of the soap film is determined by a variational principle $\delta S = 0$ where $S = \int \mathrm{d}A$. Parametrise the area with ξ^1 and ξ^2.

$$S = \int \mathrm{d}A(\xi^1, \xi^2)$$

Call the soap surface Σ ("world surface" — this term is used when one coordinate is time), and three-dimensional space is called the *target space*. Consider functions $x^m(\xi^1, \xi^2)$ that for each set of values of (ξ^1, ξ^2) we get a point in the target space.

$$x^\mu(\xi): \quad \Sigma \rightarrow T$$

Now, let (ξ^1, ξ^2) take values in some range of parameters

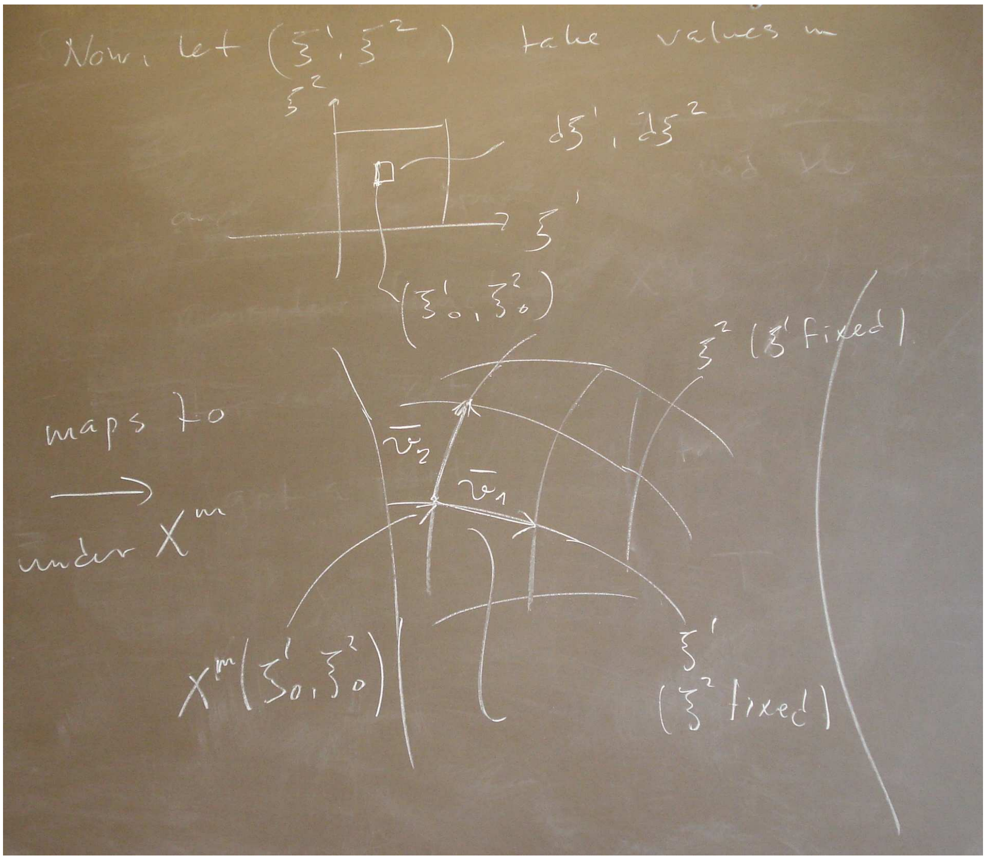

Figure 4.

$$\begin{cases} \boldsymbol{v}_1 \equiv \frac{\partial \boldsymbol{x}}{\partial \xi^1} \mathrm{d}\xi^1 \\ \boldsymbol{v}_2 \equiv \frac{\partial \boldsymbol{x}}{\partial \xi^2} \mathrm{d}\xi^2 \end{cases}$$

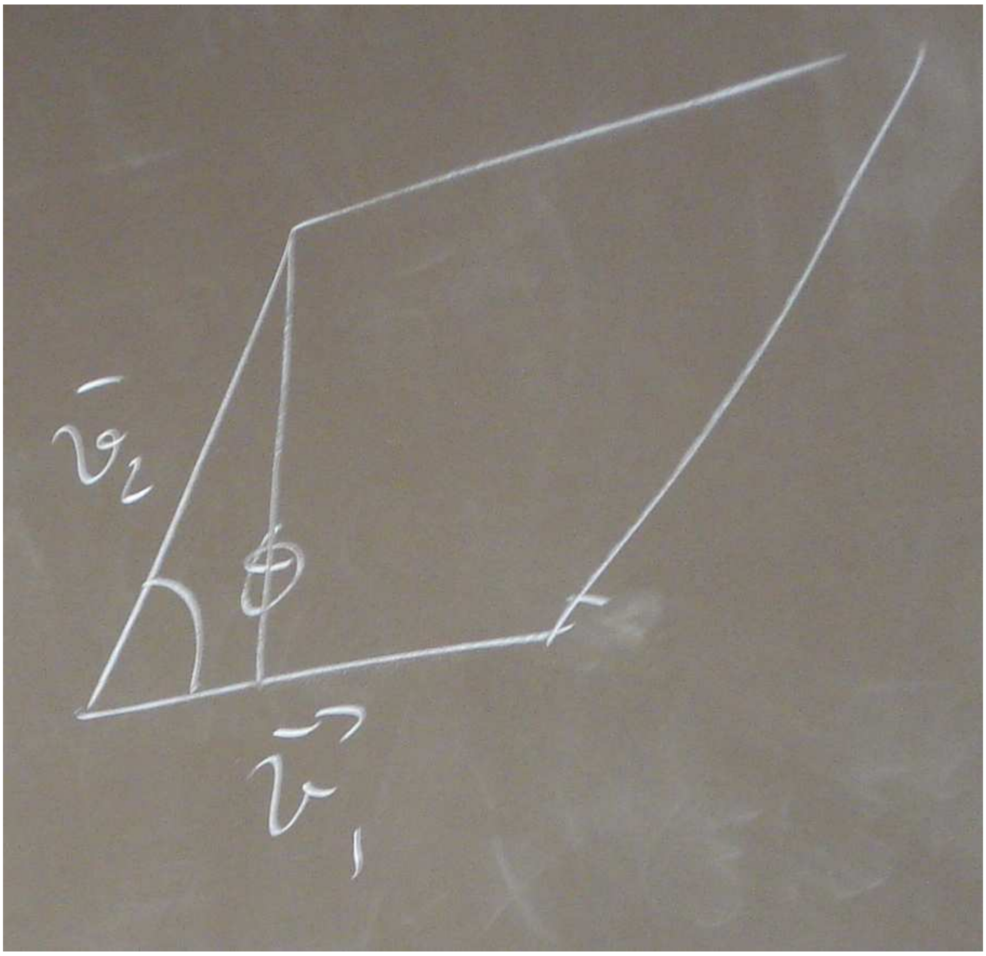

Figure 5.

$$\mathrm{d}A = |\boldsymbol{v}_1|\,|\boldsymbol{v}_2|\sin\theta = |\boldsymbol{v}_1|\,|\boldsymbol{v}_2|\sqrt{1-\cos^2\theta} = \left(|\boldsymbol{v}_1|^2\,|\boldsymbol{v}_2|^2 - \cos^2\theta\,|\boldsymbol{v}_1|^2\,|\boldsymbol{v}_2|^2\right)^{1/2} =$$

$$= \Big(|\boldsymbol{v}_1|^2\,|\boldsymbol{v}_2|^2 - (\boldsymbol{v}_1\cdot\boldsymbol{v}_2)^2\Big)^{1/2} =$$

$$= \left(\left(\frac{\partial x^m}{\partial \xi^1}\frac{\partial x^m}{\partial \xi^1}\right)\left(\frac{\partial x^n}{\partial \xi^2}\frac{\partial x^n}{\partial \xi^2}\right) - \left(\frac{\partial x^m}{\partial \xi^1}\frac{\partial x^m}{\partial \xi^2}\right)^2\right)^{1/2}\mathrm{d}\xi^1\,\mathrm{d}\xi^2$$

§ 6.2: Reparametrisation invariance

Define the metric:

$$g_{ij}(\xi) = \frac{\partial x^m}{\partial \xi^i}\frac{\partial x^n}{\partial \xi^j}\delta_{mn}, \quad \text{where } \xi^i = \left(\xi^1, \xi^2\right)$$

$$\Rightarrow \mathrm{d}A = \mathrm{d}\xi^1\,\mathrm{d}\xi^2\sqrt{\det g_{ij}(\xi)}$$

$g_{ij}(\xi)$ is the metric on the soap film, but it is *induced* via the embedding $x^m(\xi)$ from the flat metric in the background target space. g_{ij} is the pull-back of δ_{mn}.

Example: $S^2 \in \mathbb{R}^3$:

$$\mathrm{d}s^2\big|_{S^2} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big|_{S^2} = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Einstein: $R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = 0$

$$S = \frac{1}{\kappa^2}\int \underbrace{\mathrm{d}^4x\,\sqrt{-g}}_{\text{invariant}}\,R$$