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Last time: Relativistic physics in any dimension. Light-cone coordinates, light-cone stuff. Extra dimensions $\rightarrow$ compact extra dimensions (here we talked about fundamental domains, $\mathbb{R}/\mathbb{Z}_2$, $\mathbb{C}/\mathbb{Z}_n$). Quantum mechanics on a cylinder, which led to the concept of "low energy theory". To go from the string to a field theory — that's a typical low energy limit, and the cylinder example illustrates what a low energy limit can mean.

Chapter 3: Electromagnetism and gravity in various dimensions

The important points here are the following:

- Maxwell's equations in various dimensions,
- Gravity and Newton's constant,
- Compactification and Newton's constant, and then we'll talk about large extra dimensions.

§ 3.1: Classical electromagnetism

We write Maxwell's equations in Heaviside–Lorentz units:

$$\underbrace{\begin{cases} \nabla\times\boldsymbol{E} &= -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \\ \nabla\cdot\boldsymbol{B} &= 0, \end{cases}}_{\text{equations without sources}} \qquad \underbrace{\begin{cases} \nabla\cdot\boldsymbol{E} &= \rho, \\ \nabla\times\boldsymbol{B} &= \frac{1}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}. \end{cases}}_{\text{equations \textit{with} sources}}$$

There is a huge difference between the equations without sources and those with sources.

$$\begin{array}{ll} \rho\text{:} & \text{charge density,}\quad \text{esu}/L^3 \\ \boldsymbol{J}\text{:} & \text{current density,}\quad \text{esu}\times v/\text{volume} = \text{esu}/L^2 T \end{array}$$

Note that the $\boldsymbol{E}$ field and the $\boldsymbol{B}$ field have the same unit:

$$[\boldsymbol{E}] = [\boldsymbol{B}] = \frac{\text{esu}}{L^2}$$

Recall,

$$\text{charge } q \sim \int_{S^2} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{\sigma}$$

Lorentz force law

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)$$

Now, $\nabla\cdot\boldsymbol{B} = 0 \Rightarrow \boldsymbol{B} = \nabla\times\boldsymbol{A}$, for some $\boldsymbol{A}$. $\boldsymbol{A}$ is called a vector potential. Introduce the epsilon-symbol ε^{ijk}, with $\varepsilon^{123}\equiv 1$. We write $\boldsymbol{x} = x^i = (x, y, z)$.

$$B^i = \varepsilon^{ijk}\partial_j A_k, \qquad \left(\Rightarrow B^1 = \partial_2 A_3 - \partial_3 A_2;\quad \partial_2 \equiv \frac{\partial}{\partial x^2} \text{ and } \partial_3 \equiv \frac{\partial}{\partial x^3}\right)$$

$$\nabla\cdot\boldsymbol{B} = \partial_i B^i = 0.$$

$$\nabla\cdot\boldsymbol{B} = \varepsilon^{ijk}\underbrace{\partial_i\partial_j}_{=\partial_j\partial_i} A_k \equiv 0 \quad \text{(Make sure that you understand this.)}$$

The curl notation is very tied to three dimensions, but the ε notation works in any dimension. For the electric field:

$$\nabla \times E = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{c}\nabla \times \left(\frac{\partial \boldsymbol{A}}{\partial t}\right)$$

$$\Rightarrow \quad \nabla \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$

If the curl vanishes, the field is conservative and can be expressed in terms of a scalar potential:

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \equiv -\nabla \phi$$

Thus $\boldsymbol{E} = -\nabla \phi - \frac{1}{c}\dot{\boldsymbol{A}}$, where $\dot{\boldsymbol{A}} \equiv \partial_t \boldsymbol{A}$.

Note: In classical physics, only $\boldsymbol{E}$ and $\boldsymbol{B}$ are relevant. In quantum mechanics also the potentials $(\phi, \boldsymbol{A})$ are important!

Are ϕ and $\boldsymbol{A}$ well-defined given $\boldsymbol{E}$ and $\boldsymbol{B}$? (This is an extremely important question.)

Answer: ϕ and $\boldsymbol{A}$ are not unique, given $\boldsymbol{E}$ and $\boldsymbol{B}$, since $\boldsymbol{E}$ and $\boldsymbol{B}$ will not change when we replace

$$\begin{cases} \boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\varepsilon \\ \phi \rightarrow \phi - \frac{1}{c}\dot{\varepsilon} \end{cases}$$

Check: $\boldsymbol{B} = \nabla \times \boldsymbol{A} \rightarrow \boldsymbol{B}' = \nabla \times \boldsymbol{A}' = \boldsymbol{B} + \nabla \times \nabla\phi = \boldsymbol{B}$, since the curl of a gradient always vanishes. This is called *gauge invariance*. The whole Standard Model of particle physics is based upon gauge invariance. In this instance we talk about the symmetry group U(1).

We say that two gauge related sets of $(\phi, \boldsymbol{A})$ are physically equivalent.

• It can happen on spaces with non-trivial closed loops. If you have a cylinder, there is a trivial loop on the side, but a non trivial loop that goes around the cylinder.

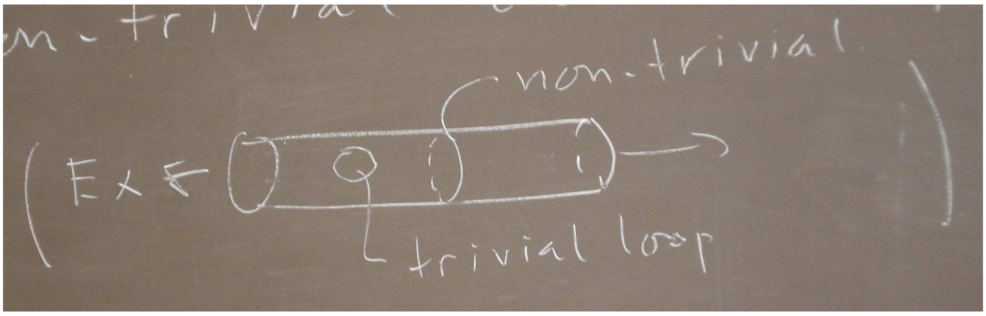

Figure 1.

Then two sets of $(\phi, \boldsymbol{A})$ with the same $(\boldsymbol{E}, \boldsymbol{B})$ can still be physically equivalent. Wilson loop.

• On complicated spaces we might need many sets of $(\phi, \boldsymbol{A})$ to produce a given $(\boldsymbol{E}, \boldsymbol{B})$ — different sets in different patches. If you have a sphere, you might have one set on the north hemisphere and an other on the south hemisphere. There must be an overlap of the two patches where there is a gauge transformation between $(\phi, \boldsymbol{A})$ and $(\phi', \boldsymbol{A}')$. Then these potentials are called *admissible* (Swedish: *tillåtna*).

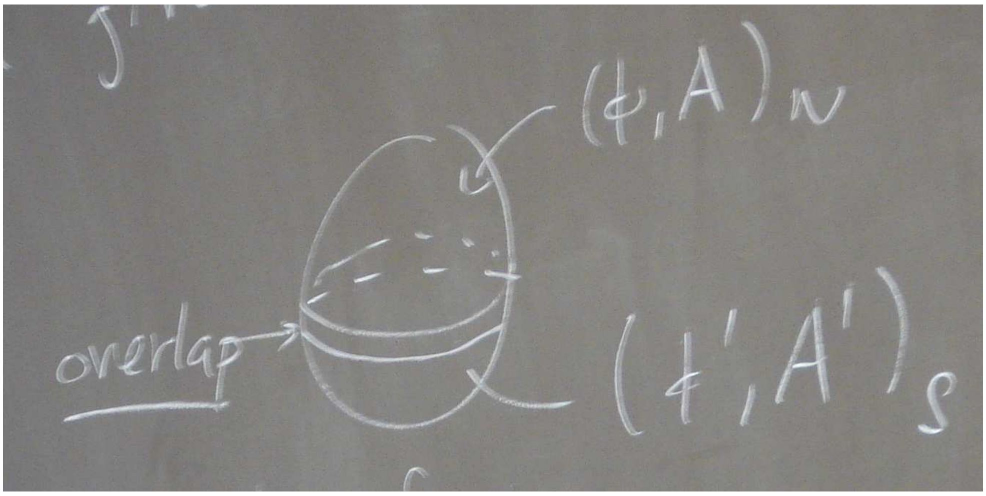

Figure 2.

§ 3.2: Electromagnetism in $2+1$ dimensions

Can we eliminate in a consistent way the effect of the space direction z? Consider the equation:

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

$\nabla = (\partial_x, \partial_y, \cancel{\partial_z})$, $\boldsymbol{j} = (j_x, j_y, 0)$, $\boldsymbol{E} = (E_x, E_y, 0)$, $\quad \Rightarrow \boldsymbol{B} = (0, 0, B_z)$.

Electromagnetism in 2 + 1 dimensions is described by (E_x, E_y, B), dropping the index on B_z because it is the only component of the $\boldsymbol{B}$ field.

§ 3.3: Manifestly relativistic electromagnetism

(The word manifest means that it should be obvious that the equations are relativistically invariant.)

$$A^\mu = (\phi, \boldsymbol{A}) \text{ is a 4-vector. Note that } A_\mu = (-\phi, \boldsymbol{A}).$$

What about $\boldsymbol{E}$ and $\boldsymbol{B}$? That's six components, in total. Let's try

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and then we can check it, by computing e.g.

$$F_{0i} = \partial_0 A_i - \partial_i(-\phi) = \frac{1}{c}\partial_t A_i + \partial_i \phi = -E_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B^k$$

- Gauge invariance is obvious: $A_\mu \to A_\mu + \partial_\mu \varepsilon$ gives $\partial_\mu \partial_\nu \varepsilon - \partial_\nu \partial_\mu \varepsilon \equiv 0$.
- Source-free equations: $\partial_{[\mu} F_{\nu\rho]} = 0$. *Bianchi identity* follows from the definition of $F_{\mu\nu}$!
- Source equations: $\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu$, where $j^\mu = (c\rho, j^i)$

These equations are valid in any dimension!

$$\boxed{\begin{array}{c} \partial_{[\mu} F_{\nu\rho]} = 0 \\ \partial_\nu F^{\mu\nu} = \dfrac{1}{c} j^\mu \end{array}}$$

This is what we mean by Maxwell's equations in arbitrary dimension.

§ 3.4: Spheres and balls in d dimensions

Notation: We write the sphere as S^d, and the ball as B^d.

$$S^{d-1} = \partial B^d$$

where ∂ in this context means "the boundary". We need formulae for the volumes:

$$I = \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

Proof.

$$I^2 = \int \mathrm{d}x \int \mathrm{d}y\,\mathrm{e}^{-x^2-y^2} = \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix} = \int r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{e}^{-r^2} = 2\pi \int \mathrm{d}r\,r\,\mathrm{e}^{-r^2} = 2\pi \left[\frac{\mathrm{e}^{-r^2}}{-2}\right]_0^\infty = \pi$$

□

Repeat this for any d: $r^2 = x_1^2 + x_2^2 + \cdots + x_d^2$

$$\Rightarrow \int \mathrm{d}^d x\,\mathrm{e}^{-r^2} = \pi^{d/2}$$

We write $\mathrm{d}^d x = \mathrm{d}r\,\mathrm{d}\Big(\mathrm{Vol}\,S_{(r)}^{d-1}\Big)$. This is the volume of a shell element.

$$\mathrm{d}^d x = \mathrm{d}r\,r^{d-1}\,\mathrm{Vol}\Big(S_{\text{unit}}^{d-1}\Big)$$

$$\int \mathrm{d}^d x\,\mathrm{e}^{-r^2} = \int_0^\infty \mathrm{d}r\,r^{d-1}\,e^{-r^2}\,\mathrm{Vol}(S_{\text{unit}}^{d-1})$$

We can carry the r integration out. Set $t = r^2, \mathrm{d}t = 2r\,\mathrm{d}r$.

$$\Rightarrow \int_0^\infty \mathrm{d}r\,r^{d-1}\,e^{-r^2} = \frac{1}{2}\int_0^\infty \mathrm{d}t\,t^{d/2-1}\,\mathrm{e}^{-t}$$

Noting that

$$\Gamma(x) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}\,t^{x-1}, \quad (x > 0)$$

$$\Rightarrow \quad \mathrm{Vol}(S_{\text{unit}}^{d-1}) = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$$

Recall that

$$\begin{cases} \Gamma(x+1) = x\,\Gamma(x) \\ \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi} \\ \Gamma(1) = 1 \end{cases}$$

For integers $n \in \mathbb{Z}$ we have $\Gamma(n+1) = n!$

Examples:

$$\begin{aligned} d = 2 &\;\Rightarrow\; \mathrm{Vol}(S^1) = 2\pi \\ d = 3 &\;\Rightarrow\; \mathrm{Vol}(S^2) = 4\pi \end{aligned}$$

§ 3.5: Electric fields in higher dimensions

Here $d =$ the number of space dimensions. Maxwell's equations: $\partial_\nu F^{\mu\nu} = \frac{1}{c}\,j^\mu$.

$$\mu = 0: \quad \partial_i\,F^{0i} = \frac{1}{c}\,j^0 \quad \Rightarrow \quad \partial_i E^i = \frac{1}{c}(c\,\rho) \quad \Rightarrow \quad \partial_i\,E^i = \rho$$

In d space dimensions $[\rho] = \mathrm{esu}/L^d$ and $[E] = \mathrm{esu}/L^{d-1}$.

Also, in any dimension: $\boldsymbol{F} = q\,\boldsymbol{E}$. $[\boldsymbol{F}] = \mathrm{N}$ in any dimension.

$$\mathrm{esu} = \sqrt{\frac{M\,L^d}{T^2}} = L^{d-3} \text{ in natural units.}$$

In three spatial dimensions, charge is dimensionless, as we said before.

Next: integrate Gauss's law over $B^d_{(R)}$ (the d dimensional ball with radius R):

$$\int_{B^d_{(R)}} \mathrm{d}(\mathrm{Vol})\, \nabla\cdot\boldsymbol{E} = \int_{B^d_{(R)}} \mathrm{d}(\mathrm{Vol})\, \rho = q\Big|_{\text{inside } B^d_{(R)}}$$

The left hand side can be written (by the divergence theorem / Gauss's theorem):

$$\int_{B^d_{(R)}} \mathrm{d}(\mathrm{Vol})\, \nabla\cdot\boldsymbol{E} = \int_{S^{d-1}_{(R)}} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{\sigma} = \begin{pmatrix} \text{flux through the} \\ \text{surface } S^{d-1}_{(R)} \end{pmatrix}$$

Proof. Split B^d into small cubes and compute the flux through the surfaces of such a d-dimensional cube. First:

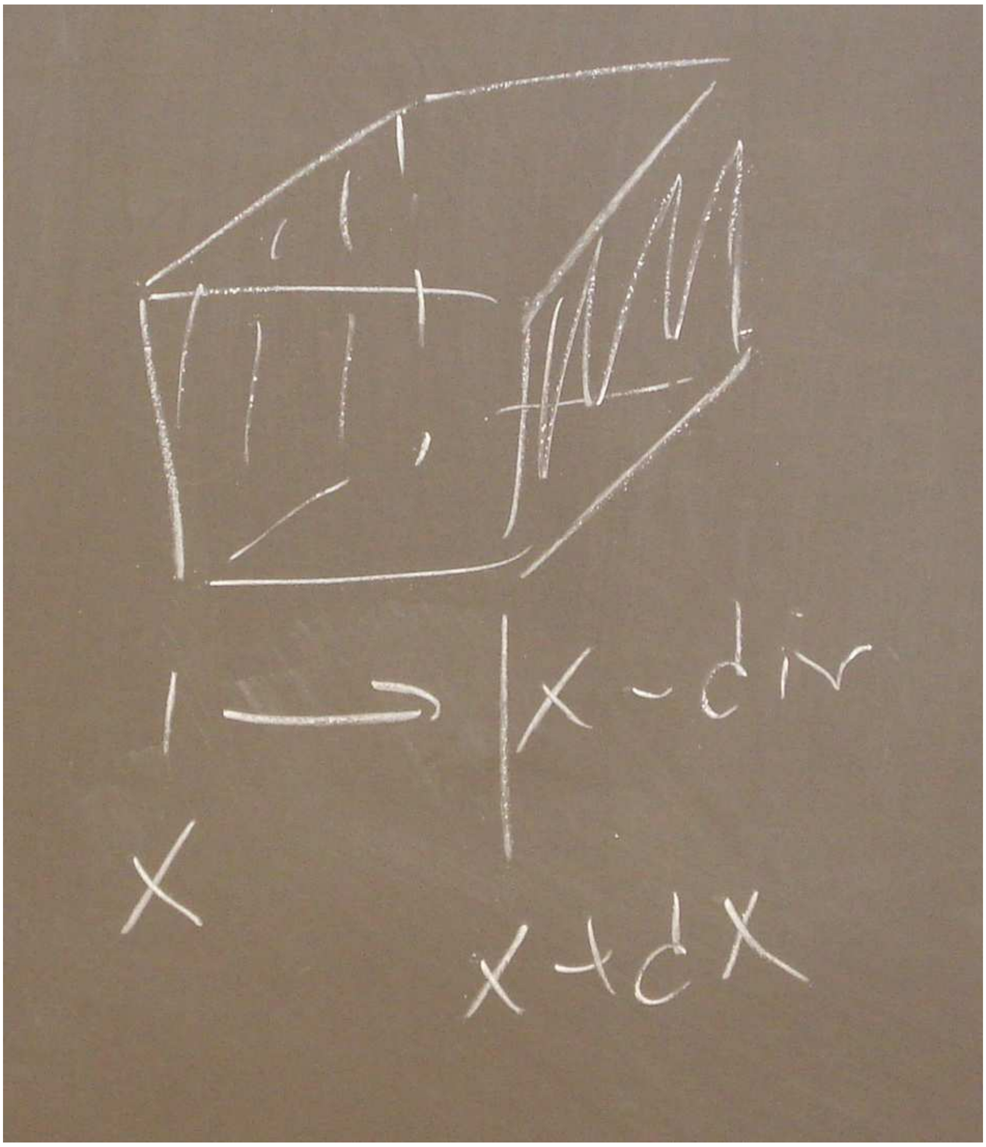

Figure 3.

$$\mathrm{flux}|_{x\text{-direction}} = (E_x(x+\mathrm{d}x, y, \ldots) - E_x(x, y, \ldots))\,\mathrm{d}y\,\mathrm{d}z \cdots =$$

$$= \partial_x E_x(x, y, \ldots)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Adding this up in all directions, we get the theorem. $\square$

$$q = \int_{S^{d-1}_{(R)}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\sigma} = \int_{S^{d-1}_{(R)}} E_r (\mathrm{d}\boldsymbol{\sigma})_r = E_r(r)\, r^{d-1} \underbrace{\mathrm{Vol}(S^{d-1}_{\text{unit}})}_{\text{known!}}$$

$$\Rightarrow E_r(r) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{q}{r^{d-1}}$$

§ 3.6: Gravitation and Planck length

In string theory we have only one parameter l_s, the length of the string. How is this related to Newton's constant? (To be discussed later.) How is Newton's constant related to the compact dimensions? (To be discussed presently.)

In special relativity $-\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$. In general relativity

$$-\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu.$$

Compute the curvature $R^\mu{}_{\nu\rho\sigma}$ (compare $F_{\mu\nu}$ in electromagnetism: both are field strengths in the same sense). Only if the space is flat is $R^\mu{}_{\nu\rho\sigma} = 0$!

Maxwell $\partial_\nu F^{\mu\nu} = 0$ "field equations"

$$\Rightarrow \quad \square A^\mu - \partial^\mu (\partial_\nu A^\nu) = 0, \quad \text{where } \square = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

wave equation $\Rightarrow$ light. Is there a similar equation in general relativity? It is horribly non-linear, so we have to linearise:

$$\begin{array}{l} g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \end{array}, \quad \text{where } h^{\mu\nu} \equiv \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}.$$

The gauge symmetry is coordinate invariance. Einstein's equations for $h_{\mu\nu}$:

$$\square h^{\mu\nu} - (\partial^\mu \partial_\rho h^{\nu\rho} + \partial^\nu \partial_\rho h^{\mu\rho}) - \partial^\mu \partial^\nu h = 0, \quad \text{where } h = h^\sigma{}_\sigma$$

We get gravitational waves.

$$\begin{cases} \delta A_\mu = \partial_\mu \varepsilon & \text{in electromagnetism,} \\ \delta h^{\mu\nu} = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu & \text{in general relativity.} \end{cases}$$

Relation between Newton's constant G and the Planck length in various dimensions.

Newton $F = G\, M_1\, M_2 / r^2$ (spacetime dimensions $= D$. Here $D = d + 1 = 3 + 1$.)

$$[G] = \frac{\mathrm{N\, m^2}}{\mathrm{kg}} = \frac{L^3}{M\, T} = L^2 \text{ in natural units}$$

$\sqrt{G} \sim$ length. So, define the Planck length:

$$l_{\mathrm{Pl}} = \sqrt{\frac{\hbar\, G}{c^3}} \approx 1.6 \times 10^{-35} \text{ m.}$$

$$t_{\mathrm{Pl}} = \frac{l_{\mathrm{Pl}}}{c} = 5.4 \times 10^{-44}\, \text{s.}$$

$$m_{\mathrm{Pl}} = \sqrt{\frac{\hbar\, c}{G}} = 2.17 \times 10^{-5}\, \text{g}$$

The Planck mass is not to be compared to things on a human scale, but to things such as the weight of the electron.

$$E_{\mathrm{Pl}} = m_{\mathrm{Pl}}\, c^2 = 1.2 \times 10^{19}\,\mathrm{GeV}$$

This is the typical scale of quantum gravity!

§ 3.7: Gravitational potential

$$\left.\begin{array}{lll} \text{Electromagnetism} & \boldsymbol{F} = q\,\boldsymbol{E}, & \boldsymbol{E} = -\nabla\phi \\ \text{Gravity} & \boldsymbol{F} = m\,\boldsymbol{g}, & \boldsymbol{g} = -\nabla V_g \end{array}\right\} \quad \text{true in all dimensions}$$

where V_g is the gravitational potential. Units:

$$[V_g] = \frac{\text{energy}}{\text{mass}}, \text{ independent of the dimensionality of the space-time.}$$

In any dimension:

$$\nabla^2 V_g = 4\pi\, G^{(D)} \rho_m$$

The left hand side is dimension-independent, $[\rho_m] = \mathrm{kg}/L^{D-1}$

$$\Rightarrow \quad \left[G^{(D)}\right] = [G]\, L^{D-4}$$

§ 3.8: The Planck length in various dimensions

$l_{\mathrm{Pl}}^{(D)}$ is the unique quantity with dimension length expressible only in terms of $\hbar$, c and $G^{(D)}$. In natural units $[G^{(D)}] = L^{D-2}$.

$$\Rightarrow \quad l_{\mathrm{Pl}}^{(D)} = \left(\frac{\hbar\, G^{(D)}}{c^3}\right)^{\frac{1}{D-2}}$$

$$\boxed{G^{(D)} = \left(l_{\mathrm{Pl}}^{(D)}\right)^{D-2} \frac{G^{(4)}}{\left(l_{\mathrm{Pl}}^{(4)}\right)^2}}$$

This is an important equation.