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Chapter 2 (Zwiebach, 2009): Special relativity and extra dimensions

Important points:

- There are three basic units: kg, m, s. For general units: M, L, T.
- There are two fundamental constants: c (m/s), $\hbar$ ($\mathrm{N\,m/s} = \mathrm{kg\,m^2/s}$).

Also

- Light-cone coordinates.
- Extra dimensions (Lorentz invariant compact extra dimensions).

§ 2.1: Units and parameters.

Units are arbitrary, in the sense that you can use m, cm, mm, etc. Observables (that is, measurable quantities) are dimensionless, i.e., numbers. For instance, ratios of masses: m_2/m_1, m_3/m_1, etc.

SI system:

$$1\,\mathrm{s} = 9192631770 \text{ periods of radiation from a certain transition in } {}^{133}\mathtt{Cs}.$$

$$1\,\mathrm{m} = \left(\frac{c}{299\,792\,458}\right)\mathrm{s}, \quad \text{where } c = 299\,792\,458\,\mathrm{m/s}.$$

$$1\,\mathrm{kg} = \text{the mass of the prototype in France.}$$

That is a very ugly definition.

What about the definition of charge?

Gaussian units: Here the unit is $[q] = \mathrm{esu}$, defined such that the Coulomb law takes the form

$$F = \frac{q_1\,q_2}{r^2} \quad \Rightarrow \quad (1\,\mathrm{esu})^2 = 10^{-5}\,\mathrm{N}\left(10^{-2}\,\mathrm{m}\right)^2 = 10^{-9}\,\mathrm{N\,m^2}$$

$$\mathrm{esu} = \frac{\mathrm{kg}^{1/2}\,\mathrm{m}^{3/2}}{\mathrm{s}}$$

In SI-units: $[q] = \mathrm{C}$

$$F = \frac{q_1\,q_2}{4\pi\varepsilon_0\,r^2}$$

Def: $1\,\mathrm{C} = 1\,\mathrm{A}\cdot 1\,\mathrm{s}$, where the ampère (A) is defined in terms of the current in two wires at a prescribed distance from each other, with a prescribed force between them.

Natural units: $\hbar = c = 1, \quad L = T = M^{-1}$.

$$1\,\mathrm{esu} = \frac{\mathrm{kg}^{1/2}\,\mathrm{m}^{3/2}}{\mathrm{s}} \quad \to \quad \frac{L^{3/2}}{L^{1/2}\,L} = 1$$

Charge is dimensionless in four dimensions.

§ 2.2: Intervals and Lorentz transformations

$$x^\mu = (\underbrace{ct}_{=x^0}, x, y, z)$$

Interval:

$$-\Delta s^2 = -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 = \eta_{\mu\nu}\,\Delta x^\mu\,\Delta x^\nu$$

$$\begin{array}{ll} \Delta s^2 > 0 & \text{timelike} \\ \Delta s^2 = 0 & \text{lightlike} \\ \Delta s^2 < 0 & \text{spacelike} \end{array}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

Provided that we take $\eta_{\mu\nu} = \eta_{\nu\mu}$ this can be read off from the Δs^2-equation.

Lorentz transformations

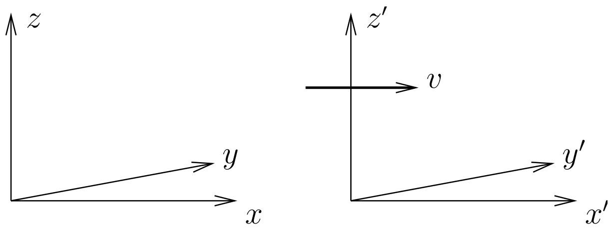

Figure 1. Two coordinate systems in standard configuration.

$$x'^\mu = L^\mu{}_\nu\, x^\nu$$

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Invariant interval $\eta_{\mu\nu}\, x^\mu\, x^\nu = \eta_{\rho\sigma}\, x'^\rho\, x'^\sigma$. Inserting $x'^\rho = L^\rho{}_\nu\, x^\nu$ we find

$$\eta_{\mu\nu} = \eta_{\rho\sigma} L^\rho{}_\mu\, L^\sigma{}_\nu\,.$$

The set of matrices L satisfying this equation make up the Lorentz group SO(1, 3). The Lorentz group leaves $\eta_{\mu\nu}$ invariant (tensor). $\eta'_{\mu\nu} = \eta_{\mu\nu}$.

Writing the equation without indices, we find $\eta = L^T \eta\, L$. Compare to SO(4): $\mathbf{1} = L^T L$.

§ 2.3: Light-cone coordinates

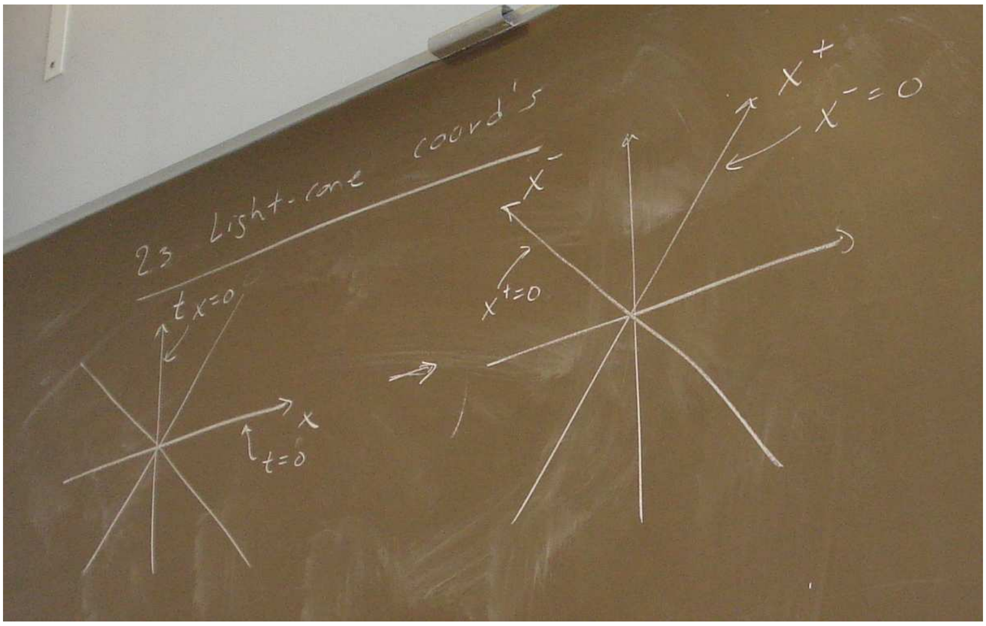

Figure 2.

Define:

$$\begin{cases} x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right) \\ x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) \end{cases}$$

But then $2\,x^+ x^- = (x^0)^2 - (x^1)^2$, so

$$\eta_{\mu\nu}\, x^\mu\, x^\mu = x_\mu\, x^\mu = -\,2\,x^+ x^- + (x^2)^2 + (x^3)^2$$

$$\Rightarrow x_\mu\, x^\mu = \hat{\eta}_{\mu\nu}\,\hat{x}^\mu\,\hat{x}^\nu$$

with $\hat{x}^\mu \equiv \left(x^+, x^-, x^2, x^3\right)$ and

$$\hat{\eta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compared to the ordinary metric, we exchange in the upper left hand corner

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The question is: what is the *light-cone time*? By definition we pick x^+ as the light-cone time, and hence x^- is light-cone space coordinate.

Note: $\hat{x}_\mu = \hat{\eta}_{\mu\nu}\,\hat{x}^\nu \Rightarrow \hat{x}_- = -\,\hat{x}^+$. We always keept the $+$ and $-$ upstairs!

Example: Particle with velocity $\boldsymbol{v} = (v, 0, 0)$.

$$\begin{cases} x^0 = c\,t \\ x^1 = v\,t = \beta\,x^0 \\ x^2 = 0 \\ x^3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right) = \frac{1}{\sqrt{2}}\,x^0(1+\beta) \\ x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) = \frac{1}{\sqrt{2}}\,x^0(1-\beta) \end{cases}$$

Velocity in $\hat{x}$-coordinates:

$$\hat{\boldsymbol{v}} = (\hat{v}^-, \hat{v}^y, \hat{v}^z) = (\hat{v}^-, 0, 0), \text{with } \hat{v}^- = \frac{\mathrm{d}\hat{x}^-}{\mathrm{d}x^+} = \frac{1-\beta}{1+\beta}$$

$$\hat{\boldsymbol{v}} = \left(\frac{1-\beta}{1+\beta}, 0, 0\right)$$

Consider a photon with velocity c in the x-direction: $\beta = 1$.

$$\hat{\boldsymbol{v}} = (0, 0, 0)$$

For a photon with velocity c in direction $-x$: $\beta = -1$:

$$\hat{\boldsymbol{v}} = (\infty, 0, 0)$$

All velocities in light cone coordinates go $0 < \hat{v}^- < \infty$: not bounded by c. This is a non-relativistic property!

Question: Are the light-cone coordinates obtainable from a Lorentz transformation? No! This is a coordinate transformation that does not leave the metric invariant (the condition for being a Lorentz transformation).

§ 2.4: Energy and momentum

x^μ is a 4-vector: $x'^\mu = L^\mu{}_\nu\, x^\nu$

$$\Rightarrow \quad \mathrm{d}x^\mu \text{ is a 4-vector} \quad \Rightarrow \quad U^\mu = c\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s} \quad \Rightarrow \quad P^\mu = m\,c\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$$

Note:

$$\mathrm{d}s = \sqrt{\mathrm{d}s^2} = \sqrt{c^2\,\mathrm{d}t^2 - (\mathrm{d}\boldsymbol{x})^2} = c\,\mathrm{d}t\,\sqrt{1-\beta^2}$$

$$\Rightarrow P^\mu = \gamma\,m(c, \boldsymbol{v})$$

§ 2.5: Light-cone 4-momentum

We have x^+ being light-cone time. What is light-cone energy? Time and energy are related, and the relation is given by the Schrödinger equation:

$$\mathrm{i}\hbar\,\frac{\partial\psi}{\partial t} = \frac{E}{c}\,\psi$$

In light-cone coordinates we have

$$\mathrm{i}\hbar\,\frac{\partial\psi}{\partial x^{+}} = \frac{E_{\text{l.c.}}}{c}\,\psi$$

To answer this consider a solution to the Schrödinger euqation:

$$\psi \sim \mathrm{e}^{\mathrm{i}\,p\cdot x/\hbar}$$

$$p\cdot x = p_\mu x^\mu = p_0\,x^0 + p_x\,x + p_y\,y + p_z\,z$$

(Note $p_0 x^0 = -\,p^0 x^0$)

$$\Rightarrow \quad \frac{E}{c} = -\,p_0 = p^0$$

In light-cone coordinates

$$\frac{E_{\text{l.c.}}}{c} = -\,p_{+} = p^{-}$$

so $H_{\text{l.c.}} = c\,p^{-}$.

§ 2.6: Extra dimensions

More time directions are not easy to make sense of (but people are still trying). But extra space directions are nice and easy, and needed in string theory.

Example: $d=6$ Lorentz symmetry SO(1, 5).

$$x'^M = L^M{}_N\,X^N$$

with $M=(0,1,\ldots,5)=(\mu,m)$ where μ takes $(0,1,2,3)$ and m takes $(4,5)$.

$$\eta_{MN} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & \delta_{mn} \end{array}\right)$$

$$L^M{}_N = \left(\begin{array}{c|c} \underset{\mathrm{SO}(1,3)}{L^\mu{}_\nu} & L^\mu{}_n \\ \hline L^m{}_\nu & L^m{}_n \end{array}\right)$$

The transformation mixes ordinary space-time coordinates with the new directions.

§ 2.7: Compact extra dimensions

In high-energy, so far, we can only detect down to $\sim 10^{-18} - 10^{-20}$ m $\sim 10\,\text{TeV}$ (LHC).

$$\hbar\,c \approx 0,2\,\text{GeV}\cdot 1\text{fm}$$

$(1\,\mathrm{fm} = 1\,\mathrm{F} = 10^{-15}\,\mathrm{m})$. But large extra dimensions are possible in Brane worlds in string theory. Large extra dimensions are in this context something on the scale of $\sim 10\,\mu\mathrm{m} = 10^{-5}\,\mathrm{m}$.

What do we mean exactly?

Consider a two-dimensional world that is a cylinder.

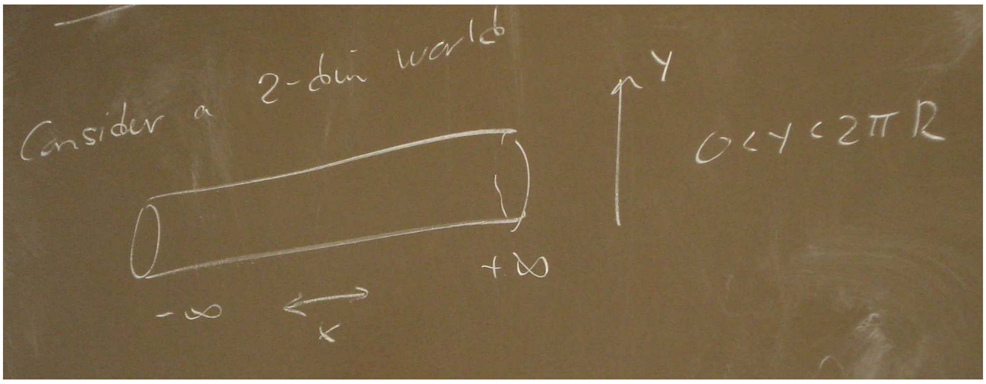

Figure 3. Cylinder world.

Circle dimensions can be viewed as follows:

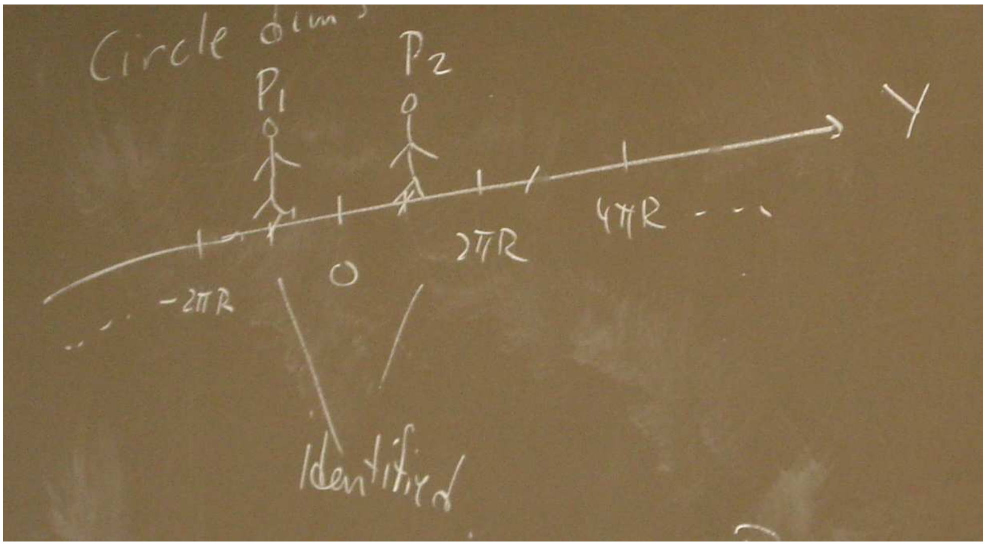

Figure 4. Divide into pieces, and identify points.

We write this as $P_1 \sim P_2$ or $P(y) \sim P(y + 2\pi n\, R)$ or loosely $y \sim y + 2\pi R$.

Now: the fundamental domain is one set of independent points.

Definition: The fundamental domain

- inside where all points are different,
- boundaries are identified,

- all outside points can be mapped to the inside.

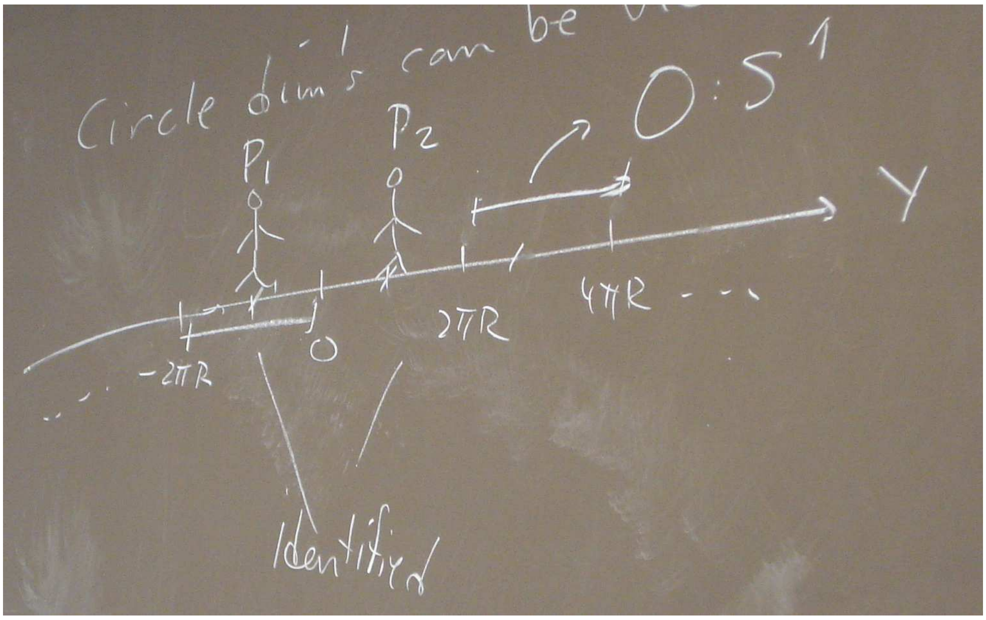

Figure 5. The fundamental domain in the above example is the circle S^1.

Coordinates on a circle

Two options:

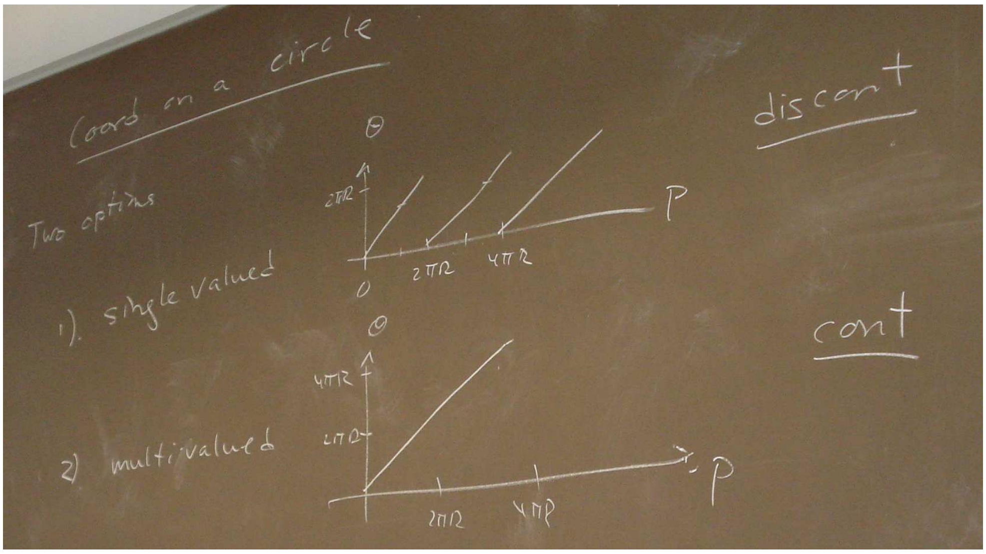

Figure 6. Either single valued and discontinuous or multivalued and continuous.

Often multivalued coordinates are easier to use.

As an aside: if θ were well-defined (which is is *not* above), then

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta = 0 \text{ (not the case here).}$$

Compare with a contour integral around the origin:

$$\oint_0 \mathrm{d}z = 0 \text{ since } z \text{ is well-defined in } \mathbb{R}.$$

Note $z = R\,\mathrm{e}^{\mathrm{i}\theta} \Rightarrow \mathrm{d}\theta \sim \mathrm{d}z/z$.

EXAMPLE: 2-torus.

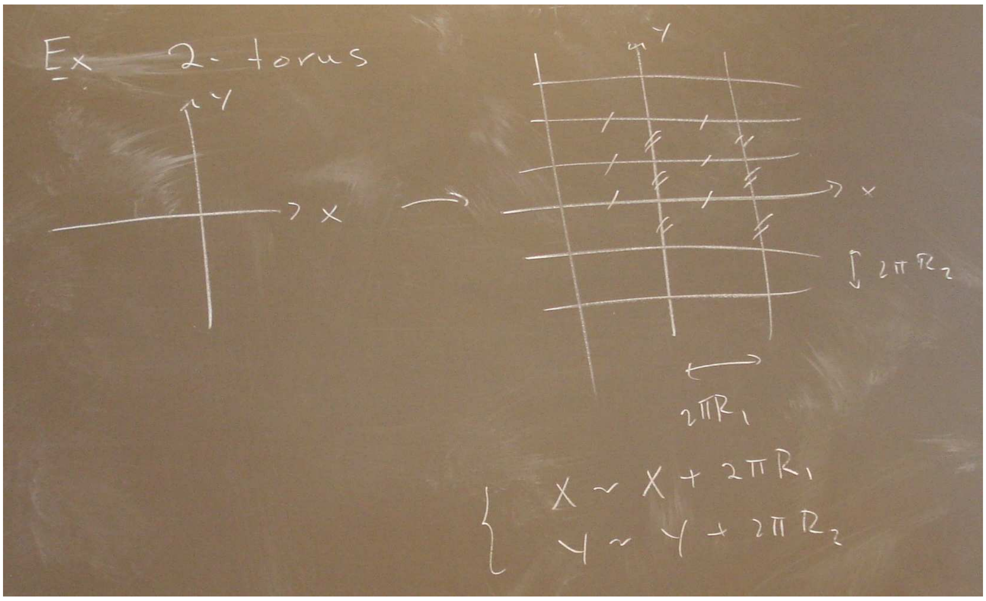

Figure 7. Identifying points to get a 2-torus.

$$\begin{cases} x \sim x + 2\pi R_1 & \Rightarrow \quad (x, y) \sim (x + 2\pi R_1, y) \\ y \sim y + 2\pi R_2 & \Rightarrow \quad (x, y) \sim (x, y + 2\pi R_2) \end{cases}$$

The fundamental domain $= T^2$ (the 2-torus).

HERE: equaivalence relations without fixpoints.

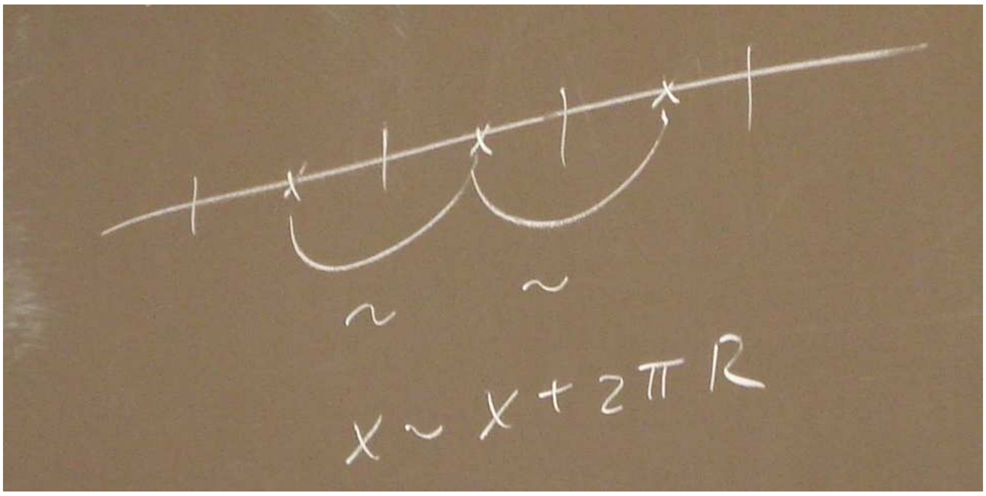

Figure 8.

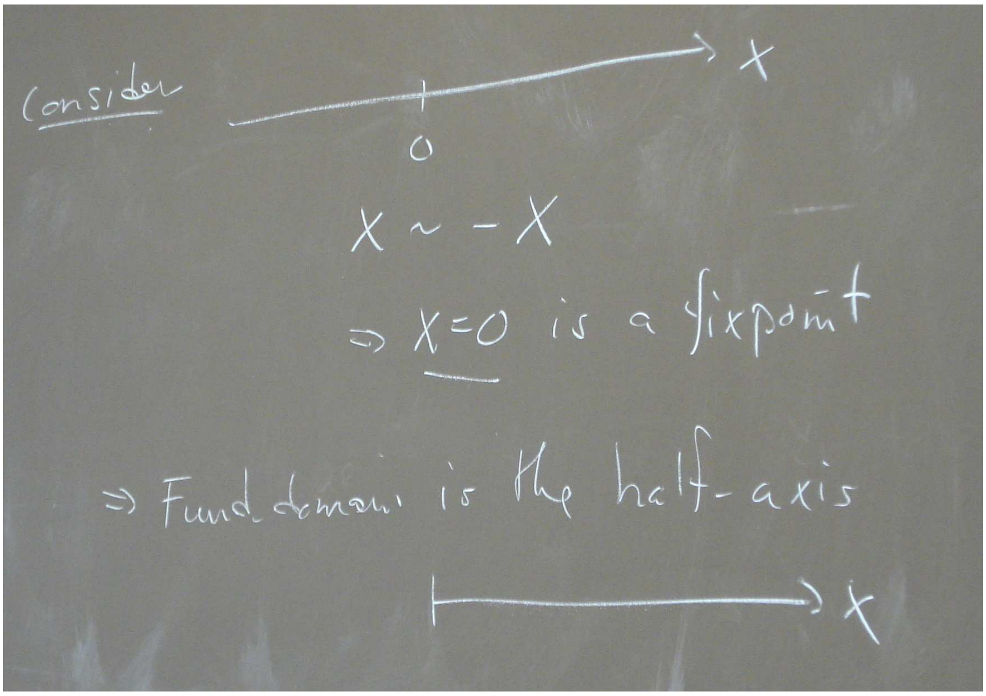

Figure 9. $x \sim -x \Rightarrow x = 0$ is a fixpoint. The fundamental domain is the half-axis. This is an orbifold (the torus, by contrast, is a manifold: smooth).

$\mathbb{R}/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{1, -1\}$.

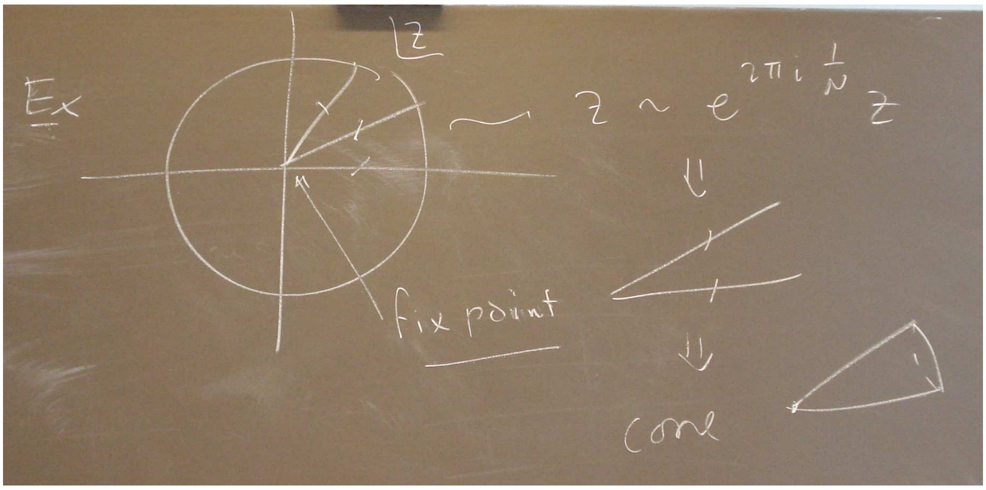

Figure 10. Making a cone by identifying points.

$\mathbb{C}/\mathbb{Z}_n, n \in \mathbb{Z}$. This notation tells us that the complex plane has been divided into pieces in some way. Normally this is a cone.

§ 2.8–2.9: Quantum mechanics and extra dimensions

First: square well

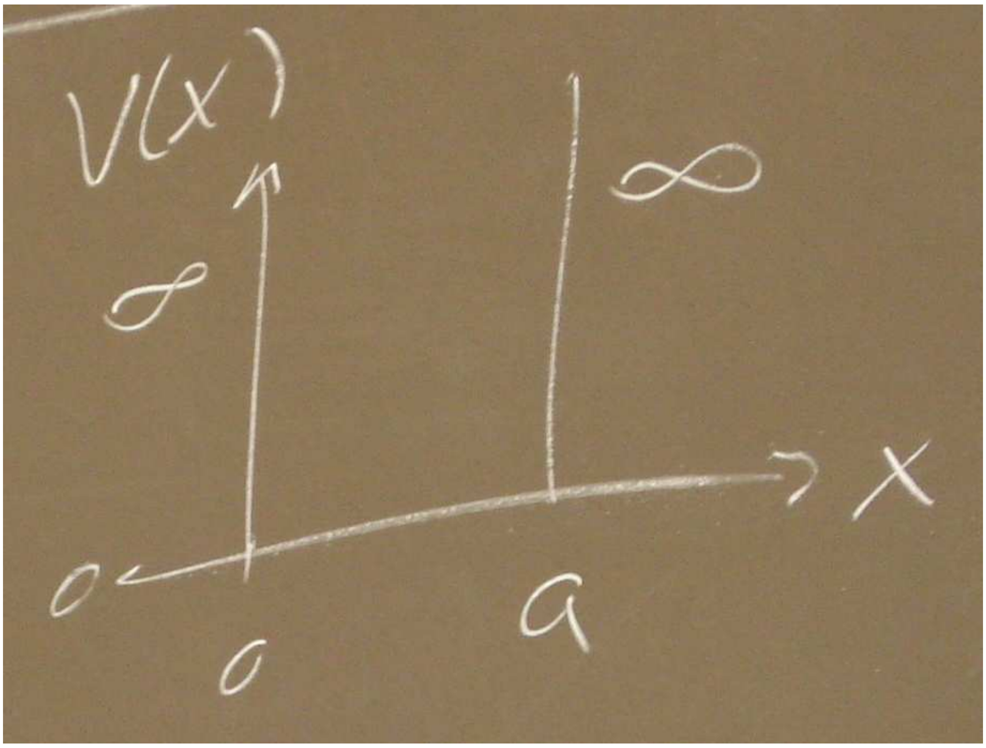

Figure 11. Square well potential

Solve

$$-\frac{k^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\,\psi=E\,\psi,\quad \psi=0 \text{ outside } [0,a]$$

$$\psi(x)\sim\sin\frac{k\,\pi\,x}{a},\quad k=1,2,3,\ldots$$

$$\Rightarrow E_k=\frac{\hbar^2}{2\,m}\left(\frac{k\,\pi}{a}\right)^2$$

Now add a second circle direction but use the same potential V (independent of $y\in S^1$).

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)=E\,\Psi$$

Separate variables: $\Psi(x,y)=\psi(x)\,\phi(x)$

$$\Rightarrow-\frac{\hbar^2}{2m}\left(\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{\phi}\frac{\partial^2\phi}{\partial y^2}\right)=E$$

Call the first term E_1 and the second E_2. E_1 becomes E_k from above and E_2 becomes

$$E_l=a_l\sin\frac{l\,y}{R}+b_l\cos\frac{l\,y}{R}$$

$$\Rightarrow E_{k,l}=\frac{\hbar^2}{2\,m}\left(\left(\frac{k\,\pi}{a}\right)^2+\left(\frac{l}{R}\right)^2\right)$$

View a as the scale of our physics and R as the scale of the compact dimension. Discuss $R \ll a$.

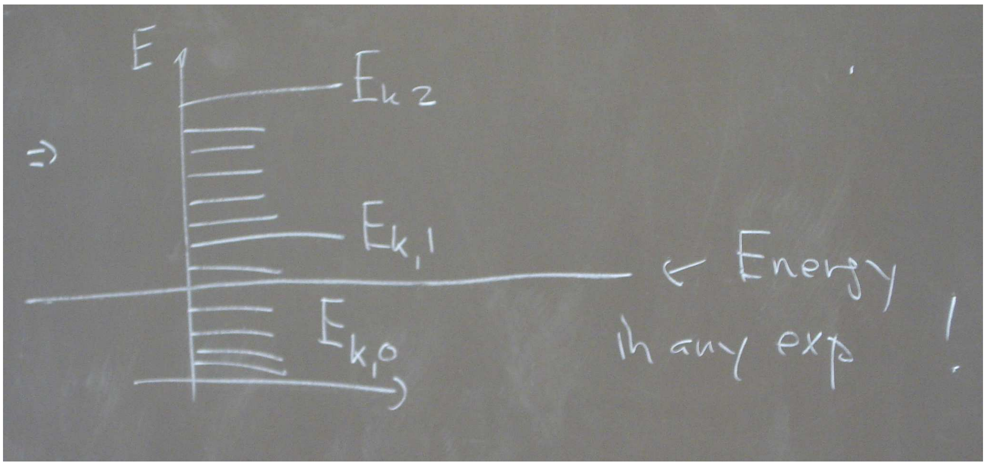

Figure 12.

We will not see the extra dimension if we do experiments below the scale of $E_{k,1}$.

In string theory we have the string scale (its length) $\sim l_s$.

$$\left\{ \begin{array}{l} l_s < R < a \\ R < l_s < a \end{array} \right. : \quad \text{there is a duality transformation between these two.}$$