Homework 3
Quantum Mechanics

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3.1 Two-dimensional harmonic oscillator

"Consider a particle in two-dimensional harmonic harmonic potential, so the Hamiltonian is given by

$$H = \frac{1}{2}k(x^2 + y^2) + \frac{1}{2m}(p_x^2 + p_y^2)$$

- Write this as two independent operators using creation and annhilation [*sic*] operators in the two axes x and y.
- Explain in a sentence or two why how [*sic*] the Hilbert space is in fact a tensor product
- Write down a formula for the energy spectrum.

Now express the *angular momentum operator* L corresponding to angular momentum around the z-axis in terms of the creation and annihilation operators a_x and a_y and their complex conjugate; how would this angular momentum operator generalize to three dimensions?

- show explicitly that L and H commute in this representation
- Compute explicitly the eigenvalues and eigenvectors in the harmonic oscillator for the six lowest energy eigenstates.
- Using your knowledge of the ground state of the single harmonic oscillator, write down $\langle r|\psi\rangle$ where $|\psi\rangle$ is the lowest energy eigenstate with angular momentum 1 and $|r\rangle$ represents the eigenket of the position operator.

This problem can fairly simply be generalized to three dimensions when you can find simultaneous eigenvalues of H, L^2 and L_z but is sufficiently messy that I won't ask you to work it out."

Two independent operators. We can write H as $H = H_x + H_y$, where

$$H_x = \frac{1}{2}kx^2 + \frac{1}{2m}p_x^2; \quad H_y = \frac{1}{2}ky^2 + \frac{1}{2m}p_y^2$$

We define ω as $\omega = \sqrt{k/m}$, and replace k above with $m\omega^2$. Then we define the annihilation operators a_x and a_y as

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\mathrm{i}p_x}{m\omega}\right); \quad a_x^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\mathrm{i}p_x}{m\omega}\right)$$

where $a_x^\dagger$ is called the creation operator. a_y and $a_y^\dagger$ are defined analogously.

Now the operators take the form

$$H_x = \hbar\omega(a_x^\dagger a_x - \tfrac{1}{2}), \quad H_y = \hbar\omega(a_y^\dagger a_y - \tfrac{1}{2}),$$
$$H = \hbar\omega(a_x^\dagger a_x - \tfrac{1}{2}) + \hbar\omega(a_y^\dagger a_y - \tfrac{1}{2}).$$

A tensor product. When the Hamiltonian can be divided into two parts, $H = H_x + H_2$ as above, we can solve the problems corresponding to H_1 and H_2 separately. The solutions of H_1 will be in a Hilbert space with kets $|\psi_1\rangle$, and the solutions of H_2 will be in a Hilbert space with kets $|\psi_2\rangle$. The set of solutions $|\psi\rangle$ of the entire problem with Hamiltonian H will then be spanned by $|\psi_1\rangle \otimes |\psi_2\rangle$, the tensor product between the two. That's *how* it is. But we were also asked *why* this is. This is seen the easiest if we study a state $|\psi\rangle$ that is a direct tensor product of eigenstates (one can of course have linear combinations, too). But this way we see why it is reasonable:

$$H|\psi\rangle = H_1(|\psi_1\rangle \otimes |\psi_2\rangle) + H_2(|\psi_1\rangle \otimes |\psi_2\rangle) =$$
$$= (H_1|\psi_1\rangle) \otimes |\psi_2\rangle + |\psi_1\rangle \otimes (H_2|\psi_2\rangle) = E_1|\psi_1\rangle \otimes |\psi_2\rangle + E_2|\psi_1\rangle \otimes |\psi_2\rangle =$$
$$= (E_1 + E_2)|\psi\rangle$$

This is what we would expect on physical grounds, and the tensor product fulfils our expectations.

A formula for the energy spectrum. Let $|n\rangle \otimes |m\rangle = |n,m\rangle$ be the state where the x oscillator is in eigenstate $|n\rangle_x$ and the y oscillator in the state $|m\rangle_y$. With $H_x|n\rangle_x = E_n|n\rangle_x$ where $E_n = \hbar\omega(n + \frac{1}{2})$ (and analogously for y), we have

$$H|n,m\rangle = E_{n,m}|n,m\rangle = (E_n + E_m)|n,m\rangle$$

and thus $E_{n,m} = \hbar\omega(n+m+1)$ is the energy spectrum of the two-dimensional harmonic oscillator.

The angular momentum around the z axis is given by

$$L = (\boldsymbol{r} \times \boldsymbol{p})_z = xp_y - yp_x.$$

(There is no ambiguity in the order, when taking the classical expression and making operators of it. x commutes with p_y, and y commutes with p_x. In three dimensions we would get an L_x, an L_y and an L_z as the components of $\boldsymbol{r} \times \boldsymbol{p}$.)

Now, we use

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a_x + a_x^\dagger\right), \quad p_x = \mathrm{i}\sqrt{\frac{\hbar m\omega}{2}} \left(-a_x + a_x^\dagger\right)$$

and similarly for y and p_y. This gives us

$$xp_y = \frac{\mathrm{i}\hbar}{2} \left(a_x + a_x^\dagger\right) \left(-a_y + a_y^\dagger\right) = \frac{\mathrm{i}\hbar}{2} \left(-a_x a_y + a_x a_y^\dagger - a_x^\dagger a_y + a_x^\dagger a_y^\dagger\right)$$

$$yp_x = \frac{\mathrm{i}\hbar}{2} \left(-a_y a_x + a_y a_x^\dagger - a_y^\dagger a_x + a_y^\dagger a_x^\dagger\right)$$

The a_x and the a_y commute, so we get $L = \mathrm{i}\hbar \left(a_x a_y^\dagger - a_x^\dagger a_y\right)$.

Now we should show that L and H commute:

$$[L, H] = [L, H_x + H_y] = [L, H_x] + [L, H_y] =$$

$$= \mathrm{i}\hbar \left(a_y^\dagger [a_x, H_x] - a_y [a_x^\dagger, H_x]\right) + \mathrm{i}\hbar \left(a_x [a_y^\dagger, H_y] - a_x^\dagger [a_y, H_y]\right).$$

$$[a_x, H_x] = [a_x, \hbar\omega(a_x^\dagger a_x + \tfrac{1}{2})] = \hbar\omega [a_x, a_x^\dagger] a_x + \hbar\omega a_x^\dagger [a_x, a_x] = \hbar\omega a_x$$

$$[a_x^\dagger, H_x] = \hbar\omega [a_x^\dagger, a_x^\dagger] a_x + \hbar\omega a_x^\dagger [a_x^\dagger, a_x] = -\hbar\omega a_x^\dagger$$

$$\implies [L, H] = \mathrm{i}\hbar^2\omega a_y^\dagger a_x + \mathrm{i}\hbar^2\omega a_y a_x^\dagger - \mathrm{i}\hbar^2\omega a_x a_y^\dagger - \mathrm{i}\hbar^2\omega a_x^\dagger a_y = 0$$

as desired.

The six lowest energy states. They are:

$$|n{,}m\rangle \in \{|0{,}0\rangle, |1{,}0\rangle, |0{,}1\rangle, |1{,}1\rangle, |1{,}2\rangle, |2{,}1\rangle\}$$

That's the eigenvectors. The eigenvalues are $\hbar\omega$, $2\hbar\omega$, $2\hbar\omega$, $3\hbar\omega$, $4\hbar\omega$ and $4\hbar\omega$, respectively, according to the above energy formula.

The lowest energy eigenstate with angular momentum 1 is the state $|\psi\rangle$ so that $L\,|\psi\rangle = \hbar \cdot 1\,|\psi\rangle$. We see that $L\,|0{,}0\rangle = 0$, so we take the next higher energy states:

$$L\,|n{,}m\rangle = \mathrm{i}\hbar\sqrt{n}\sqrt{m+1}\,|n-1{,}m+1\rangle - \sqrt{n+1}\sqrt{m}\,|n+1{,}m-1\rangle$$

$$L\,|1,0\rangle = \mathrm{i}\hbar\,|0,1\rangle$$

$$L\,|0,1\rangle = -\mathrm{i}\hbar\,|1,0\rangle$$

A linear combination of these will do:

$$L(\alpha\,|1{,}0\rangle + \beta\,|0{,}1\rangle) = \mathrm{i}\hbar(-\beta\,|1,0\rangle + \alpha\,|0{,}1\rangle) = \hbar(-\mathrm{i}\beta\,|1,0\rangle + \mathrm{i}\alpha\,|0{,}1\rangle)$$

This is an eigenstate if $\alpha = -\mathrm{i}\beta$ and $\beta = \mathrm{i}\alpha$. We can take $\alpha = 1/\sqrt{2}$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|1{,}0\rangle + \frac{\mathrm{i}}{\sqrt{2}}\,|0{,}1\rangle$$

Next, we are asked to find $\langle r|\psi\rangle$, where $|r\rangle = |x\rangle \otimes |y\rangle$.

$$\langle r|\psi\rangle = \frac{1}{\sqrt{2}}\,\langle x|1\rangle_x\,\langle y|0\rangle_y + \frac{\mathrm{i}}{\sqrt{2}}\,\langle x|0\rangle_x\,\langle y|1\rangle_y$$

For a harmonic oscillator, we have in general

$$\langle x|0\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

where $x_0 = \sqrt{\hbar/m\omega}$. We have

$$\langle x|1\rangle = \frac{1}{\sqrt{2}x_0}\left(x - x_0^2\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle = \sqrt{2}\frac{x}{x_0}\,\langle x|0\rangle$$

$$\begin{aligned}\langle r|\psi\rangle &= \frac{1}{\sqrt{2}}\sqrt{2}\frac{x}{x_0}\,\langle x|0\rangle_x\,\langle y|0\rangle_y + \frac{\mathrm{i}}{\sqrt{2}}\sqrt{2}\frac{y}{x_0}\,\langle x|0\rangle_x\,\langle y|0\rangle_y = \\ &= \left\{\frac{x}{x_0^2\sqrt{\pi}} + \frac{\mathrm{i}y}{x_0^2\sqrt{\pi}}\right\}\exp\left[-\frac{1}{2}\frac{(x^2+y^2)}{x_0^2}\right]\end{aligned}$$

So the answer is

$$\boxed{\langle r|\psi\rangle = \frac{m\omega(x+\mathrm{i}y)}{\hbar\sqrt{\pi}}\exp\left[-\frac{m\omega(x^2+y^2)}{2\hbar}\right]}$$

3.2 Sakurai 2.11

> "Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by
>
> $$\exp\left(\frac{-\mathrm{i}pa}{\hbar}\right)|0\rangle\,,$$
>
> where p is the momentum operator and a is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$." (Sakurai, problem 2.11, page 145.)

First, what do we expect? We recognise the exponential above to be exactly the translation operator $\mathscr{T}$ of Sakurai's equation (1.6.36). It produces a spatial translation of the state $|0\rangle$ by a length a in the positive x direction. What does that give us? Sakurai calls it a *coherent state* (section 2.3, page 97) — a superposition of energy eigenstates that closely imitates the classical oscillator; a wave packet that bounces back and forth without spreading in shape.

The operator $x(t)$ is given by Sakurai's equation (2.3.45a):

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t.$$

The state $|\psi\rangle$ we are studying can be written

$$|\psi\rangle = \exp\left(\frac{-\mathrm{i}p(0)a}{\hbar}\right)|0\rangle\,,$$

where $|0\rangle$ is understood to be the state ket of the system ground state at time $t = 0$. (The base kets $|n\rangle$ would be moving in the Heisenberg picture, but the state ket $|\psi\rangle$ remains stationary.)

I will also be using Sakurai's equations (2.3.25a) and (2.3.25b) in order to say

$$\langle 0|x(0)|0\rangle = \langle 0|p(0)|0\rangle = 0$$

which can be easily proved by writing the operators in terms of creation and annihilation operators; and Sakurai's equation (2.2.23a):

$$[x, F(p)] = \mathrm{i}\hbar\frac{\partial F}{\partial p}.$$

I do hope it is permissible that I take equations from Sakurai without proving them first; in any case, the proof of this latter equation is similar to the one we did in last week's homework.

$$
\begin{aligned}
\langle x\rangle = \langle\psi|x(t)|\psi\rangle &= \langle\psi|x(0)\cos\omega t|\psi\rangle + \langle\psi|\frac{p(0)}{m\omega}\sin\omega t|\psi\rangle = \\
&= \langle 0|\exp\left(\frac{+\mathrm{i}p(0)a}{\hbar}\right)x(0)\exp\left(\frac{-\mathrm{i}p(0)a}{\hbar}\right)|0\rangle\cos\omega t + \\
&+ \langle 0|\exp\left(\frac{+\mathrm{i}p(0)a}{\hbar}\right)p(0)\exp\left(\frac{-\mathrm{i}p(0)a}{\hbar}\right)|0\rangle\frac{\sin\omega t}{m\omega}
\end{aligned}
$$

Now, since $p(0)$ commutes with a function of itself, we have

$$
\exp\left(\frac{+\mathrm{i}p(0)a}{\hbar}\right)p(0)\exp\left(\frac{-\mathrm{i}p(0)a}{\hbar}\right) =
$$

$$
= \exp\left(\frac{+\mathrm{i}p(0)a}{\hbar}\right)\exp\left(\frac{-\mathrm{i}p(0)a}{\hbar}\right)p(0) = p(0).
$$

If I may use e^+ and e^- to refer to the exponentials above, we have for $x(0)$:

$$
\mathrm{e}^+x(0)\mathrm{e}^- = \mathrm{e}^+\mathrm{e}^-x(0) + \mathrm{e}^+[x(0),\mathrm{e}^-] = x(0) + \mathrm{e}^+\left(\mathrm{i}\hbar\frac{\partial\mathrm{e}^-}{\partial p(0)}\right) =
$$

$$
= x(0) + \mathrm{e}^+\left(\mathrm{i}\hbar\left(-\frac{\mathrm{i}a}{\hbar}\right)\mathrm{e}^-\right) = x(0) + a
$$

which we of course already knew, having identified e^- as the translation operator $\mathscr{T}$ at the outset.

This leaves us with

$$
\langle x\rangle = \langle 0|x(0)|0\rangle\cos\omega t + \langle 0|0\rangle\, a\cos\omega t + \langle 0|p(0)|0\rangle\frac{\sin\omega t}{m\omega} = a\cos\omega t.
$$

Thus, the answer is $\langle x\rangle = a\cos\omega t$, a solution that closely imitates the classical oscillator, just as we expected.

3.3 Sakurai 2.4

> "Let $x(t)$ be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate $[x(t), x(0)]$."

(Remark: Revision 2 of the homework still talks about the state $|0\rangle$. Since a free particle does not have a ground state, I'm going to assume that $|0\rangle$ has been left over from the old version, where problem 3.3 still talked about Sakurai 2.15. I will do, as has been stated clearly on the homepage and at a lecture, Sakurai 2.4, and do it as it was written in Sakurai — without the $|0\rangle$.)

To this end, we will need the time evolution operator of Sakurai (2.2.9):

$$\mathscr{U}(t) = \exp\left(\frac{-\mathrm{i}Ht}{\hbar}\right)$$

where H is the Hamiltonian operator and t is the time. We will also need the commutation relation of Sakurai (2.2.23a):

$$[x, F(p)] = \mathrm{i}\hbar\frac{\partial F}{\partial p}$$

In the Heisenberg picture operators, such as $x(t)$, evolve according to the equation $x(t) = \mathscr{U}^\dagger(t)x(0)\mathscr{U}(t)$. For the commutator we then have

$$[x(t), x(0)] = x(t)x(0) - x(0)x(t) = \mathscr{U}^\dagger(t)x(0)\mathscr{U}(t)x(0) - x(0)\mathscr{U}^\dagger(t)x(0)\mathscr{U}(t).$$

To simplify our notation a bit, let's drop the arguments of these operators, and let it be understood that $\mathscr{U}$ is to be evaluated at time t, and x at time 0.

$$[x(t), x(0)] = \mathscr{U}^\dagger x\mathscr{U}x - x\mathscr{U}^\dagger x\mathscr{U} = \mathscr{U}^\dagger\mathscr{U}x^2 + \mathscr{U}^\dagger[x,\mathscr{U}]x - x\mathscr{U}^\dagger\mathscr{U}x - x\mathscr{U}^\dagger[x,\mathscr{U}].$$

We know, of course, that $\mathscr{U}^\dagger\mathscr{U} = 1$, so now we only need the commutator $[x, \mathscr{U}]$ and we will have the answer $[x(t), x(0)] = \mathscr{U}^\dagger[x, \mathscr{U}]x - x\mathscr{U}^\dagger[x, \mathscr{U}]$. In order to do this we need $\mathscr{U}$, and for that we need the Hamiltonian for the free particle:

$$H = \frac{p^2}{2m},$$

where p is the momentum operator. (Which is a constant of the motion, because it commutes with H.)

$$[x, \mathscr{U}] = \left[x, \exp\left(\frac{-\mathrm{i}Ht}{\hbar}\right)\right] = \left[x, \exp\left(\frac{-\mathrm{i}p^2t}{2m\hbar}\right)\right] =$$

$$= \mathrm{i}\hbar\frac{\partial}{\partial p}\exp\left(\frac{-\mathrm{i}p^2t}{2m\hbar}\right) = \frac{p}{m}t\mathscr{U}$$

Do note that $\mathscr{U}$, being a function of p, commutes with p. That gives us

$$[x(t), x(0)] = \mathscr{U}^\dagger\mathscr{U}\frac{p}{m}tx = x\mathscr{U}^\dagger\mathscr{U}\frac{p}{m}tx = \frac{p}{m}tx - x\frac{p}{m}t =$$

$$= [p, x(0)]\,\frac{t}{m} = \frac{-\mathrm{i}\hbar t}{m}$$

So the answer is $[x(t), x(0)] = \hbar t/m\mathrm{i}$, where t is the time and m the mass of the free particle. It would perhaps be easier to show this by following the procedure in the book, writing down equations (2.2.26), (2.2.27) and (2.2.29) — or even quote the book's equation (2.2.29) directly. But I thought that would be *too* easy, that it was probably expected that we do a bit more than just look up the right equation in the book, so I have presented an alternative derivation of (2.2.29).

3.4 Galilean transformations

> "Let the unitary operator $\mathcal{G}$ be given by $\mathcal{G} = \mathrm{e}^{\mathrm{i}v(mx-pt)/\hbar}$ where v is a constant and x and p are operators. Show that G generates the galilean transformation $x \to x - vt$ and compute the result of transforming p with G."

I will be assuming that $G = \mathcal{G}$.

The transformation is given by $x' = \mathcal{G}^\dagger x \mathcal{G}$. (We are, of course, using the Heisenberg picture; otherwise we would have $x \to x$ and a change of the state kets, instead of $x \to x - vt$ as stated in the problem text.) This means that we can use the formula

$$\mathrm{e}^X Y \mathrm{e}^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots$$

which holds for any operators X and Y. In our specific case we have

$$\mathcal{G}^\dagger x \mathcal{G} = \exp\left(-\frac{\mathrm{i}v}{\hbar}(mx - pt)\right) x \exp\left(\frac{\mathrm{i}v}{\hbar}(mx - pt)\right) =$$

$$= x + \left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), x\right] + \frac{1}{2!}\left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), \left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), x\right]\right] + \cdots$$

Since $[x, x] = 0$ and $[p, x] = -\mathrm{i}\hbar$, we have

$$\left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), x\right] = \frac{\mathrm{i}vt}{\hbar}[p, x] = vt$$

and since this is a scalar, it commutes with everything, and the series terminates at $\mathcal{G}^\dagger x \mathcal{G} = x + vt$. It would seem that one of us has made a sign error.

$$\mathcal{G}^\dagger p \mathcal{G} = p + \left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), p\right] + \frac{1}{2!}\left[-\frac{\mathrm{i}v}{\hbar}(mx - pt), [\ldots, p]\right] + \cdots$$

$$\left[-\frac{\mathrm{i}v}{\hbar}(mx-pt),p\right]=-\frac{\mathrm{i}vm}{\hbar}\,[x,p]=vm$$

This is a scalar, and so commutes with everything. $\mathcal{G}^\dagger p\mathcal{G}=p+vm$.