2.1

"A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach–type measurements as follows:

a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$, with $\hat{\boldsymbol{n}}$ making an angle β in the xz-plane with respect to the z-axis.

c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?" (Sakurai, problem 1.13.)

The state after the first measurement can here be regarded as the initial state, since it is normalised. That means that we are starting out with $|+\rangle$. The filtering process can be described as

$$|+\rangle = a\,|+_n\rangle + b\,|-_n\rangle \mapsto a\,|+_n\rangle = c\,|+\rangle + d\,|-\rangle \mapsto d\,|-\rangle\,.$$

This filtering can be carried out using projection operators:

$$d\,|-\rangle = \big(\,|-\rangle\langle -|\,\big)\,\big(\,|+_n\rangle\langle +_n|\,\big)\,|+\rangle = |-\rangle\,\langle -|+_n\rangle\,\langle +_n|+\rangle$$

In the end we are after the final intensity $|d|^2$:

$$|d|^2 = \big|\,\langle -|+_n\rangle\,\big|^2\,\big|\,\langle +_n|+\rangle\,\big|^2 \tag{1}$$

In order to compute this, we need the eigenstate $|+_n\rangle$ of the operator $\boldsymbol{S}\cdot\boldsymbol{n}$:

$$\boldsymbol{S}\cdot\boldsymbol{n} = \begin{pmatrix} S_x & S_y & S_z \end{pmatrix}\begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = S_x\sin\beta + S_z\cos\beta =$$

$$= \frac{\hbar}{2}\Big[\,|+\rangle\langle -| \;+\; |-\rangle\langle +|\,\Big]\sin\beta + \frac{\hbar}{2}\Big[\,|+\rangle\langle +| \;-\; |-\rangle\langle -|\,\Big]\cos\beta \doteq$$

$$\doteq \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

We can quickly establish the eigenvalues $\pm\hbar/2$. We can find the eigenvectors with Gaussian elimination, but to do that we would really like to be able to multiply and divide with abandon — we need to consider a few special cases first, to keep the trigonometric functions away from the dangerous values 0 and 1. Specifically, we need to treat $\beta \in \{0, \pi/2, \pi\}$ separately.

For $\beta = 0$, $\boldsymbol{S}\cdot\boldsymbol{n} = S_z$ and $|+_n\rangle = |+\rangle$, $|-_n\rangle = |-\rangle$.

For $\beta = \pi/2$, $\boldsymbol{S}\cdot\boldsymbol{n} = S_x$ and $|\pm_n\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$.

For $\beta = \pi$, $\boldsymbol{S}\cdot\boldsymbol{n} = -S_z$ and $|+_n\rangle = |-\rangle$, $|-_n\rangle = |+\rangle$.

Now, for any other β:

$$\boldsymbol{S}\cdot\boldsymbol{n} - 1I \doteq \frac{\hbar}{2}\begin{pmatrix}\cos\beta - 1 & \sin\beta \\ \sin\beta & -\cos\beta - 1\end{pmatrix} \sim$$
$$\sim \begin{pmatrix}1 & \frac{\sin\beta}{\cos\beta - 1} \\ \sin\beta & -\cos\beta - 1\end{pmatrix} \sim \begin{pmatrix}1 & \frac{\sin\beta}{\cos\beta-1} \\ 0 & 0\end{pmatrix}$$

If we take $|+_n\rangle \doteq \begin{pmatrix}x_1 \\ x_2\end{pmatrix}$, this means that x_1 and x_2 obey the equation

$$(1 - \cos\beta)\, x_1 = \sin\beta \cdot x_2.$$

Using the formulae for twice an angle, we can simplify things a bit:

$$2\sin^2\frac{\beta}{2}x_1 = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}x_2$$
$$\implies \sin\frac{\beta}{2}x_1 = \cos\frac{\beta}{2}x_2$$

Combine this with the normalisation condition $|x_1|^2 + |x_2|^2$ and we immediately see a worthy candidate:

$$|+_n\rangle = \cos\frac{\beta}{2}\,|+\rangle + \sin\frac{\beta}{2}\,|-\rangle \doteq \begin{pmatrix}\cos\frac{\beta}{2} \\ \sin\frac{\beta}{2}\end{pmatrix}. \tag{2}$$

This general expression also reproduces the special cases isolated at the outset, so we may take (2) to be the desired equation for $|+_n\rangle$. There is no need to calculate the other eigenstate $|-_n\rangle$. Going back to the expression (1) for the final intensity, we see that we need the inner products:

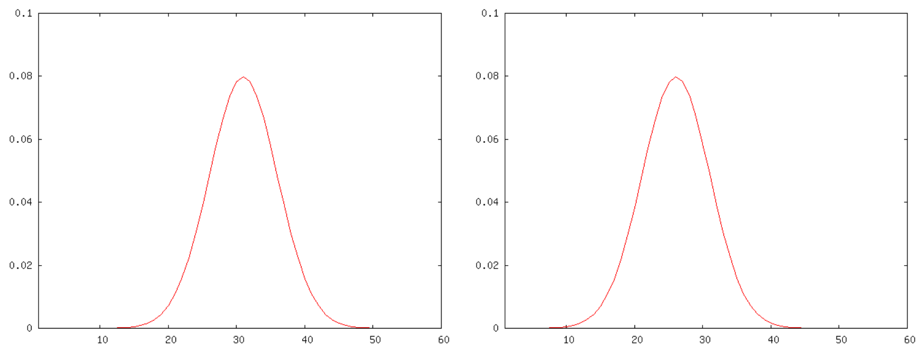
$$\langle +|+_n\rangle = \cos\frac{\beta}{2} \quad\text{and}\quad \langle -|+_n\rangle = \sin\frac{\beta}{2}$$
$$|d|^2 = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta.$$

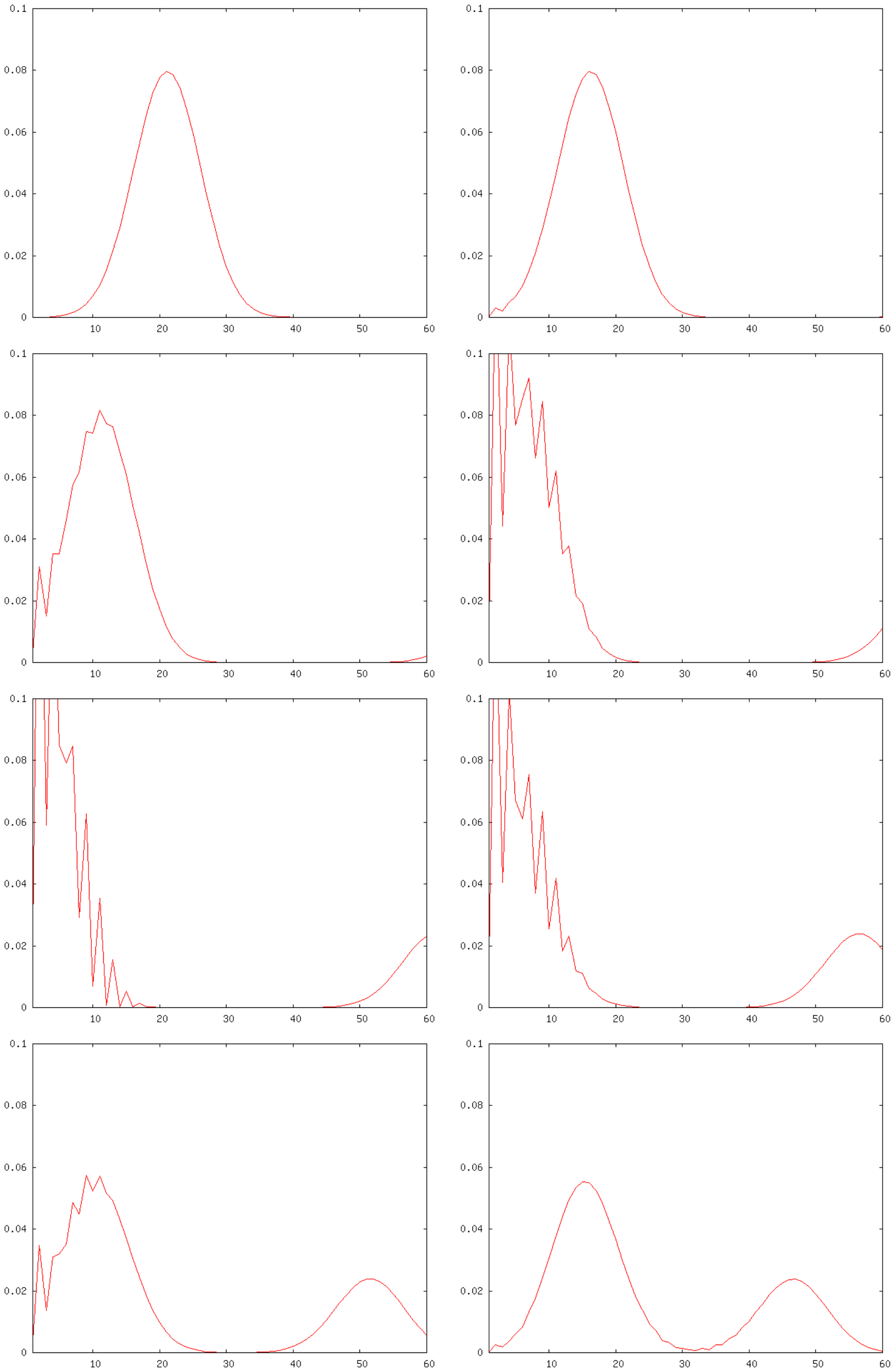
So, if the incoming intensity was unity, the final intensity will be $\frac{1}{4}\sin^2\beta$. To maximise the final intensity, the second measuring apparatus must be set at right angles with the first: $\beta = \pi/2$ produces the maximal intensity $\frac{1}{4}$.

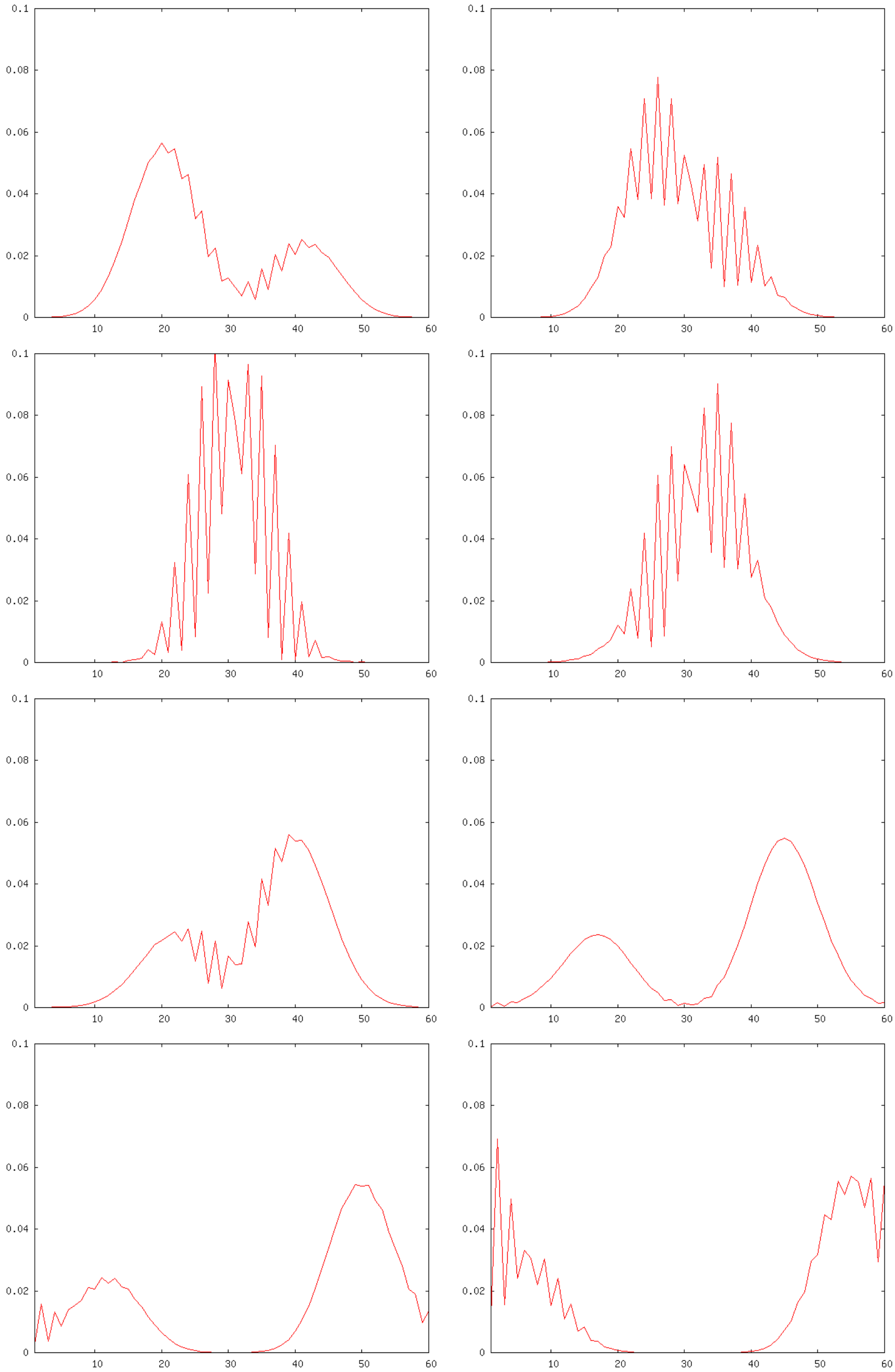
2.2

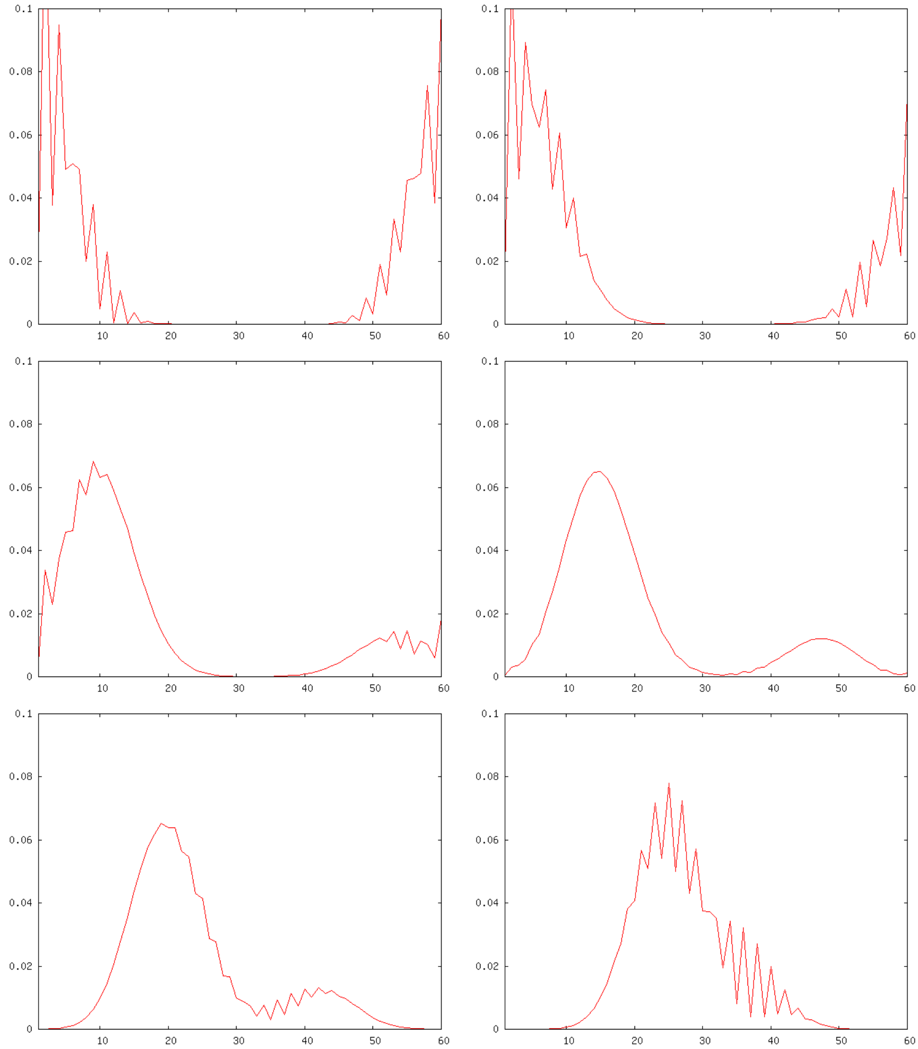
```
# -*-octave-*-

nn = 60;
id = eye ( nn );
h = - ( shift( id, 1) + shift( id, -1 ) );
h(1,1) += 3;
[u,e]  = schur(h,"a");
ev  = diag( e );
shouldbezero = h - ( u * ( e * u' ) );
deltat = .5;
ut = u * diag(  exp( i * deltat * ev ) )* u' ;

x = [0:nn-1];
psi = exp( -( ((  x - nn/2)/(nn/6) ).^2));
psi = psi ./ sqrt( psi * psi');
psi =  ( exp( 32i * pi * x / nn ) .* psi ).';

gset key off

for t = 1:120
axis([1 nn 0 .1]);
plot( abs(psi).^2 )
axis([1 nn 0 .1]);
psi = ut * psi;
eval(sprintf("print(\"-dpng\", \"fil%.3d.png\")", t ));
end

system("convert fil???.png gr.gif");
```

Here comes every fifth frame in the time evolution:




0.1
0.08
0.06
0.04
0.02
0
10
20
30
40
50
60


0.1
0.08
0.06
0.04
0.02
0
10
20
30
40
50
60



2.3

“Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0,$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0, \qquad [A_2, H] = 0.$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-

force problem $H = \boldsymbol{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$."
(Sakurai, problem 1.17.)

Since $[A_1, H] = 0$, we can diagonalise them simultaneously. Let the basis that diagonalises A_1 in this way be denoted by $\{|a^{(i)}\rangle\}$. Similarly, since $[A_2, H] = 0$ we can diagonalise A_2 and H simultaneously; let that basis be denoted by $\{|b^{(i)}\rangle\}$. Since $[A_1, A_2] \neq 0$ we know that, in general, $|a^{(i)}\rangle$ does not equal any of the $|b^{(i)}\rangle$. It will be some linear combination:

$$|a^{(i)}\rangle = \sum_j c_i^{(j)} |b^{(j)}\rangle .$$

For at least one i we will have several nonzero $c_i^{(j)}$ — otherwise we could simultaneously diagonalise A_1 and A_2, and then we would have $[A_1, A_2] = 0$. Since $|a^{(i)}\rangle$ is an eigenket of H, we have $H|a^{(i)}\rangle = h_a^{(i)} |a^{(i)}\rangle$ where $h_a^{(i)}$ is the energy eigenvalue of the eigenstate $|a^{(i)}\rangle$. Similarly, let $h_b^{(j)}$ be the energy eigenvalue of the eigenstate $|b^{(j)}\rangle$. Now, on the one hand we have

$$H|a^{(i)}\rangle = \sum_j c_i^{(j)} H |b^{(j)}\rangle = \sum_j c_i^{(j)} h_b^{(j)} |b^{(j)}\rangle ,$$

and on the other hand we have

$$H|a^{(i)}\rangle = h_a^{(i)} |a^{(i)}\rangle = \sum_j c_i^{(j)} h_a^{(i)} |b^{(j)}\rangle ,$$

which means that

$$\sum_j c_i^{(j)} h_b^{(j)} |b^{(j)}\rangle = \sum_j c_i^{(j)} h_a^{(i)} |b^{(j)}\rangle .$$

Since the $|b^{(j)}\rangle$ are linearly independent, we can equate their coefficients. Thus, provided that $c_i^{(j)} \neq 0$ we have $h_a^{(i)} = h_b^{(j)}$. But, as pointed out above, there must be at least *some* i for which $c_i^{(j)} \neq 0$ holds for several j, which means that for this i we have several $h_b^{(j)}$ taking on the same value. In other words: the energy eigenstates of H are degenerate.

2.4

"Prove the *operator* identity

$$\big[p_i, F(\boldsymbol{x})\big] = -\mathrm{i}\hbar \frac{\partial F}{\partial x_i}$$

where $F(\boldsymbol{x})$ is a function of the operator $\boldsymbol{x}$ in two ways: (1) directly from the commutation relations $[x_i, p_j] = \mathrm{i}\hbar\delta_{ij}$ and (2) by using the real space basis and evaluating $\langle \hat{\boldsymbol{x}} | [p_i, F(\boldsymbol{x})] | f \rangle$."

Using the commutation relations. Assume that the operator $F(\boldsymbol{x})$ can be written in a Taylor expansion:

$$F(\boldsymbol{x}) = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} x_1^\alpha x_2^\beta x_3^\gamma.$$

Without loss of generality, assume $i = 1$. (If it is not, we relabel the axes so that it is.) We then compute

$$[p_1, F(\boldsymbol{x})] = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} [p_1, x_1^\alpha] x_2^\beta x_3^\gamma.$$

Noting that $[a, b] = -[b, a]$, we can now call into effect the theorem of last week's homework, number 1.3:

$$[[\Omega, \Lambda], \Omega] = 0 \implies [\Omega^m, \Lambda] = m\Omega^{m-1}[\Omega, \Lambda] \text{ for operators } \Omega, \Lambda.$$

In this problem, we take $\Omega = x_1$ and $\Lambda = p_1$. $[x_1, p_1] = \mathrm{i}\hbar$, which commutes with x_1, so the requirements are fulfilled.

Thus

$$[p_1, x_1^\alpha] = \alpha x_1^{\alpha-1}[p_1, x_1] = -\mathrm{i}\hbar\alpha x_1^{\alpha-1}[p_1, x_1] = -\mathrm{i}\hbar \frac{\partial x_1^\alpha}{\partial x_1}$$

$$[p_1, F(\boldsymbol{x})] = -\mathrm{i}\hbar \sum_{\alpha,\beta,\gamma} \frac{\partial}{\partial x_1} x_1^\alpha x_2^\beta x_3^\gamma = -\mathrm{i}\hbar \frac{\partial F}{\partial x_1}$$

If we now undo the relabelling that resulted in $i = 1$, we arrive at the statement to be proved:

$$[p_i, F(\boldsymbol{x})] = -\mathrm{i}\hbar \frac{\partial F}{\partial x_i}.$$

Using the real space basis. I'll take $\boldsymbol{x}$ to be the operator, and $\hat{\boldsymbol{x}}$ to be the coordinates of the state $|\hat{\boldsymbol{x}}\rangle$. I would have found it more intuitive to have it the other way around...

In general, we have

$$\langle\beta|f(x)|\alpha\rangle = \int \mathrm{d}x' \psi_\beta^*(x') f(x') \psi_\alpha(x').$$

$$\langle\hat{\boldsymbol{x}}|[p_i, F(\boldsymbol{x})]|f\rangle = \langle\hat{\boldsymbol{x}}|p_i F(\boldsymbol{x})|f\rangle - \langle\hat{\boldsymbol{x}}|F(\boldsymbol{x}) p_i|f\rangle$$

$$\langle\hat{\boldsymbol{x}}|p_i F(\boldsymbol{x})|f\rangle = \int \mathrm{d}\boldsymbol{x}' \left(\delta(\boldsymbol{x}' - \hat{\boldsymbol{x}})\right)^* (-\mathrm{i}\hbar) \frac{\partial}{\partial x_i'} F(\boldsymbol{x}') f(\boldsymbol{x}') =$$

$$= -\mathrm{i}\hbar \int \mathrm{d}\boldsymbol{x}'\delta(\boldsymbol{x}' - \hat{\boldsymbol{x}}) \left(\frac{\partial F(\boldsymbol{x}')}{\partial x_i'} f(\boldsymbol{x}') + F(\boldsymbol{x}') \frac{\partial f(\boldsymbol{x}')}{\partial x_i'} \right) =$$

$$= -\mathrm{i}\hbar \left(\left. \frac{\partial F(\boldsymbol{x}')}{\partial x_i'} \right|_{\boldsymbol{x}'=\hat{\boldsymbol{x}}} f(\hat{\boldsymbol{x}}) + F(\hat{\boldsymbol{x}}) \left. \frac{\partial f(\boldsymbol{x}')}{\partial x_i'} \right|_{\boldsymbol{x}'=\hat{\boldsymbol{x}}} \right)$$

$$\langle \hat{\boldsymbol{x}} | F(\boldsymbol{x}) p_i | f \rangle = -\mathrm{i}\hbar F(\hat{\boldsymbol{x}}) \left. \frac{\partial f(\boldsymbol{x}')}{\partial x_i'} \right|_{\boldsymbol{x}'=\hat{\boldsymbol{x}}}$$

This gives us the following expression for the left hand side:

$$\langle \hat{\boldsymbol{x}} | [p_i, F(\boldsymbol{x})] | f \rangle = -\mathrm{i}\hbar \left. \frac{\partial F(\boldsymbol{x}')}{\partial x_i'} \right|_{\boldsymbol{x}'=\hat{\boldsymbol{x}}} f(\hat{\boldsymbol{x}})$$

Now on to the right hand side:

$$\langle \hat{\boldsymbol{x}} | \left(-\mathrm{i}\hbar \frac{\partial F}{\partial x_i} \right) | f \rangle = \int \mathrm{d}\boldsymbol{x}' \left(\delta(\boldsymbol{x}' - \hat{\boldsymbol{x}}) \right)^* \left(-\mathrm{i}\hbar \frac{\partial F(\boldsymbol{x}')}{\partial x_i'} \right) f(\boldsymbol{x}') =$$

$$= -\mathrm{i}\hbar \left. \frac{\partial F(\boldsymbol{x}')}{\partial x_i'} \right|_{\boldsymbol{x}'=\hat{\boldsymbol{x}}} f(\hat{\boldsymbol{x}})$$

We see that the expression for the left hand side above and this expression for the right hand side agree. Thus

$$[p_i, F(\boldsymbol{x})] = -\mathrm{i}\hbar \frac{\partial F}{\partial x_i}.$$

2.5

"Let the state initial state [*sic*] of a one dimensional free particle be given by $|\psi(0)\rangle$ where $\langle x|\psi(0)\rangle$ is given by the Gaussian

$$\langle x|\psi(0)\rangle = \left(\frac{2a}{\pi} \right)^{\frac{1}{4}} \mathrm{e}^{-ax^2}$$

where a is real and positive.

(a) Transform to momentum eigenstates and show that

$$\langle p|\psi(0)\rangle = \left(\frac{1}{2a\pi\hbar^2} \right)^{\frac{1}{4}} \mathrm{e}^{-\frac{p^2}{4\hbar^2 a}}$$

(b) Compute $\langle x|\psi(t)\rangle$ and show it is equal to

$$\left(\frac{2a}{\pi} \right)^{\frac{1}{4}} \frac{\mathrm{e}^{-ax^2/(1+(2\mathrm{i}\hbar at/m))}}{\sqrt{1 + (2\mathrm{i}\hbar at/m)}}$$

(c) Compute the probability density $|\langle x|\psi(t)\rangle|^2$ and show that it is a gaussian [*sic*] which spreads out in time."

Transformation to momentum eigenstates proceeds as usual

$$\langle p|\psi(0)\rangle = \int \mathrm{d}x\, \langle p|x\rangle\, \langle x|\psi(0)\rangle$$

We know that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{\mathrm{i}px}{\hbar}\right),$$

so

$$\langle p|\psi(0)\rangle = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int \mathrm{d}x \exp\left(\frac{-\mathrm{i}px}{\hbar}\right) \exp(-ax^2) =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int \mathrm{d}x \exp\left(-\frac{\mathrm{i}px}{\hbar} - ax^2\right) =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(-\left(\frac{p}{\hbar}\right)^2 /4a\right)$$

where the last step was taken using the familiar expression for Gaussian integrals with an imaginary coefficient for the x factor. Cleaning this up, we arrive at

$$\langle p|\psi(0)\rangle = \left(\frac{1}{2\pi\hbar^2 a}\right)^{\frac{1}{4}} \exp\left(-\frac{p^2}{4a\hbar^2}\right)$$

as desired.

To compute $\langle x|\psi(t)\rangle$ we need the time evolution operator $\mathscr{U}$:

$$\mathscr{U} = \exp(-\mathrm{i}Ht/\hbar),$$

where $H = \hat{p}^2/2m$ for a free particle. We then have

$$\langle x|\psi(t)\rangle = \langle x|\mathscr{U}|\psi(0)\rangle = \int \mathrm{d}p\, \langle x|\mathscr{U}|p\rangle\, \langle p|\psi(0)\rangle =$$

$$= \int \mathrm{d}p \exp\left(-\frac{\mathrm{i}p^2 t}{2m\hbar}\right) \langle x|p\rangle\, \langle p|\psi(0)\rangle =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\hbar^2 a}\right)^{\frac{1}{4}} \int \mathrm{d}p \exp\left(-\frac{\mathrm{i}p^2 t}{2m\hbar} + \frac{\mathrm{i}px}{\hbar} - \frac{p^2}{4a\hbar^2}\right) =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\hbar^2 a}\right)^{\frac{1}{4}} \int \mathrm{d}p \exp\left(-\frac{1}{2}\left(\frac{1}{2a\hbar^2} + \frac{\mathrm{i}t}{m\hbar}\right) p^2 + \frac{\mathrm{i}x}{\hbar} p\right) =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{2\pi\hbar^2 a}\right)^{\frac{1}{4}} \left(\frac{2\pi}{\frac{1}{2a\hbar^2} + \frac{\mathrm{i}t}{m\hbar}}\right)^{\frac{1}{2}} \exp\left(-\left(\frac{x}{\hbar}\right)^2 \Big/ 2\left(\frac{1}{2a\hbar^2} + \frac{\mathrm{i}t}{m\hbar}\right)\right)$$

Note that

$$\frac{1}{2a\hbar^2} + \frac{\mathrm{i}t}{m\hbar} = \frac{1 + 2\mathrm{i}\hbar at/m}{2a\hbar^2}.$$

This clears up the exponential nicely, and with some further thought we find the coefficient matches up as well:

$$\langle x|\psi(t)\rangle = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{\exp\left(-ax^2/(1 + 2\mathrm{i}\hbar at/m)\right)}{\sqrt{1 + 2\mathrm{i}\hbar at/m}}$$

As far as I can see, there is nothing in the Gaussian formula used for this that tells us which branch of the complex square root we should use — I'm not very happy with the derivations of this formula I have seen. But assuming that it is correct, this is what we get.

The probability density $|\langle x|\psi(t)\rangle|^2$ is relatively easy. If c is a complex number, $|c|^2 = |c^2| = c^*c$. We will use the first of these equalities for the square root, squaring it first, while we use the second for the exponential:

$$|\langle x|\psi(t)\rangle|^2 =$$

$$= \left|\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\right|^2 \frac{\exp\left(-ax^2/(1 - 2\mathrm{i}\hbar at/m)\right)\exp\left(-ax^2/(1 + 2\mathrm{i}\hbar at/m)\right)}{\left|\left(\sqrt{1 + 2\mathrm{i}\hbar at/m}\right)^2\right|} =$$

$$= \sqrt{\frac{2a}{\pi}} \frac{\exp\left(-ax^2\left(\frac{1}{(1-2\mathrm{i}\hbar at/m)} + \frac{1}{(1+2\mathrm{i}\hbar at/m)}\right)\right)}{\sqrt{1 + 4\hbar^2a^2t^2/m^2}} =$$

$$= \sqrt{\frac{2a}{\pi}} \frac{\exp\left(\frac{-2ax^2}{1+4\hbar^2a^2t^2/m^2}\right)}{\sqrt{1 + 4\hbar^2a^2t^2/m^2}}$$

This is a Gaussian (the only x is a x^2 with a negative coefficient, appearing in the argument of the exponential function), and it is spreading out in time. To see the latter, consider that the square of the standard deviation σ is proportional to the expression in the denominator,

$$\sigma^2 \propto 1 + 4\hbar^2a^2t^2/m^2,$$

and this increases with time.