2008–09–30

Two particles, 1 and 2.

$$\mathcal{D}(R)|\alpha_1\rangle \otimes |\alpha_2\rangle = (\mathcal{D}_1(R)|\alpha_1\rangle \otimes \mathcal{D}_2(R)|\alpha_2\rangle) = \mathcal{D}_1(R) \otimes \mathcal{D}_2(R)\,|\alpha_1\rangle \otimes |\alpha_2\rangle$$

$$\mathcal{D}_\Omega(\delta\theta) = \left(1 - \mathrm{i}\,\boldsymbol{\Omega}\cdot\boldsymbol{J}_1\,\frac{\delta\theta}{\hbar}\right) \otimes \left(1 - \mathrm{i}\,\boldsymbol{\Omega}\cdot\boldsymbol{J}_2\,\frac{\delta\theta}{\hbar}\right) = 1 - \mathrm{i}\,(\boldsymbol{J}_1 \otimes 1 + 1 \otimes \boldsymbol{J}_2)\cdot\boldsymbol{\Omega}\,\frac{\delta\theta}{\hbar} + (\delta\theta)^2$$

$$\boldsymbol{J}_{\mathrm{tot}} = \boldsymbol{J}_1 + \boldsymbol{J}_2$$

$$[\boldsymbol{J}_1, \boldsymbol{J}_2] = 0$$

$\left\{\boldsymbol{J}_1^2, \boldsymbol{J}_2^2, (\boldsymbol{J}_1)_z, (\boldsymbol{J}_2)_z\right\}$ $|j_1, j_2, m_1, m_2\rangle =$ tensor product basis.

$\left\{\boldsymbol{J}_{\mathrm{tot}}^2, (\boldsymbol{J}_{\mathrm{tot}})_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2\right\}$ all commute.

$$\boldsymbol{J}_{\mathrm{tot}}^2 = \boldsymbol{J}_{\mathrm{tot}}\cdot\boldsymbol{J}_{\mathrm{tot}} = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2\,\boldsymbol{J}_1\cdot\boldsymbol{J}_2$$

Tensor product representation	Total spin representation
$\boldsymbol{J}_1^2|j_1 j_2 m_1 m_2\rangle = j_1(j_1+1)|j_1, j_2, m_1, m_2\rangle$;	$\boldsymbol{J}_{\mathrm{tot}}^2|j_1, j_2, j, m\rangle = j(j+1)|j_1, j_2, j, m\rangle$
$\boldsymbol{J}_2^2|...\rangle = j_2(j_2+1)|...\rangle$	$J_{\mathrm{tot},z}|j_1, j_2, j, m\rangle = m\,|j_1, j_2, j, m\rangle$
$(J_1)_z|...\rangle = m_1|...\rangle$	$\boldsymbol{J}_1^2|j_1, ...\rangle = j_1(j_1+1)|j_1, j_2 ...\rangle$
$(J_2)_z ...$	$\boldsymbol{J}_2^2|j_1, ...\rangle =$

$$|j_1, j_2, j, m\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \underbrace{\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle}_{=\text{Clebsch-Gordan coefficient}}$$

$$|j_1, j_2, m_1, m_2\rangle = \sum_{j, m} |j_1, j_2, j, m\rangle\langle j_1, j_2, j, m|j_1, j_2, m_1, m_2\rangle$$

We drop the j_1, j_2 from the notation. They are now implicit.

$$\langle j, m|J_{\mathrm{tot},z}|m_1\, m_2\rangle = (m_1 + m_2)\,\langle j, m|m_1, m_2\rangle = m\,\langle j, m|m_1, m_2\rangle$$

$$m = m_1 + m_2 \ \text{ for } \langle j, m|m_1, m_2\rangle = 0$$

$$(j_{\mathrm{tot}})_{\max} = j_1 + j_2$$

$$|j_1 - j_2| \leqslant j \leqslant |j_1 + j_2|$$

Consistency check:

$$(2\,j_1 + 1)(2\,j_2 + 1) = \sum_{j=|j_1 - j_2|}^{|j_1 + j_2|} (2\,j + 1)$$

$$j_1 \otimes j_2 \quad = \quad j_1 + j_2 \quad \oplus \quad j_1 + j_2 - 1 \quad ... \quad \oplus |j_1 - j_2|$$

Example

$$2 \otimes 3 \quad = \quad 5 \oplus 4 \oplus 3 \oplus 2 \oplus 1$$

$$35 = 11 + 9 + 7 + 5 + 3$$

$$\left(\begin{array}{c} 35 \times 35\text{-matrix} \\ \text{big mess} \end{array} \right) = \left(\begin{array}{ccccc} \boxed{11 \times 11} & & & & \\ & \boxed{9 \times 9} & & & \\ & & \boxed{7 \times 7} & & \\ & & & \boxed{5 \times 5} & \\ & & & & \boxed{3 \times 3} \end{array} \right)$$

$$H = \alpha\, \boldsymbol{J}_1 \cdot \boldsymbol{J}_2 = \frac{\alpha}{2} \Big((\boldsymbol{J}_1 + \boldsymbol{J}_2)^2 - \boldsymbol{J}_1^2 - \boldsymbol{J}_2^2 \Big) = \frac{\alpha}{2} (\boldsymbol{J}_{\text{tot}} - \boldsymbol{J}_1 - \boldsymbol{J}_2)$$

Spin $1 \otimes 1$.

$$1 \otimes 1 = 2 \oplus 1 \oplus 0$$

degeneracy	j_{tot}	$\boldsymbol{J}_{\text{tot}}^2$	$\boldsymbol{J}_1^2$	$\boldsymbol{J}_2^2$	energy eigenvalues
5	2	6	2	2	α
2	1	2	2	2	$-\alpha$
1	0	0	2	2	-2α

$$1 \otimes 1 \otimes 1 = (2 \oplus 1 \oplus 0) \otimes 1 = (2 \otimes 1) \oplus (1 \otimes 1) \oplus (0 \otimes 1)$$

$$= 3 \oplus 2 \oplus 1 \quad \oplus \quad 2 \oplus 1 \oplus 0 \quad \oplus \quad 1 = 3 \oplus 2\,(2) \oplus 3(1) \oplus 0$$

$$27 = 7 + 10 + 9 + 1$$