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$$\mathcal{D}(R(\theta)) = \mathrm{e}^{-\mathrm{i}\boldsymbol{J}\cdot\boldsymbol{\Omega}\theta/\hbar} = \left(1 - \frac{\mathrm{i}\,\boldsymbol{J}\cdot\boldsymbol{\Omega}}{\eta}\,\delta\theta\right)^N, \quad \delta\theta = \frac{\theta}{N}$$

$$\lim_{N\to\infty}\left(1 - \frac{x}{N}\right)^N = \mathrm{e}^{-x}$$

$$[\boldsymbol{J}^2, J_z] = 0$$

$$\boldsymbol{J}^2|j,m\rangle = \lambda_J\,|j,m\rangle$$

$$J_z|j,m\rangle = m\,\hbar\,|j,m\rangle$$

$$J_-J_+ = \boldsymbol{J}^2 - J_z(J_z+1)$$

$$0 = J_+|j,m_{\max}\rangle \Rightarrow \lambda_j - m_{\max}(m_{\max}+1) = 0$$

$$\lambda_j - m_{\min}(m_{\min}-1) = 0$$

$$m_{\min} = -\,m_{\max}$$

$$\lambda_j = j(j+1)$$

$$m = -\,j, -\,j+1, \ldots, j-1, j$$

j is an integer or an half integer.

$$\langle j,m|J_-J_+|j,m\rangle = \hbar^2\,(j(j+1) - m(m+1))\langle j,m|j,m\rangle$$

$$J_+|j,m\rangle = \hbar\,\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle$$

$$J_-|j,m\rangle = \hbar\,\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$$

$j = 1/2$. $J^2 = j(j+1) = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$.

$|\boldsymbol{J}\,| = \sqrt{3/4}$. $J_z = -\,\frac{1}{2}, +\,\frac{1}{2}$.

Spin 1: $\boldsymbol{J}^2 = 1(1+1) = 2$.

Limit $j\to\infty$. $\boldsymbol{J}^2 \to \hbar^2\,j\,(j+1) \approx \hbar^2 j^2$.

$$[J_i, J_j] = \hbar\,\mathrm{i}\,\varepsilon_{ijk}\,J_k$$

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$$

$$[L_i, L_j] = \mathrm{i}\,\hbar\varepsilon_{ijk}\,L_k$$

$$\boldsymbol{R}^2 = x^2 + y^2 + z^2$$

$$\left[\boldsymbol{R}^2, L^2\right] = 0, \quad \left[\boldsymbol{R}^2, L_z\right] = 0$$

$$|\boldsymbol{R}^2, \boldsymbol{L}^2, L_z\rangle \quad \text{Spherical harmonics}$$

$$|p\rangle \quad \text{Fourier transforms}$$

How does $\boldsymbol{J}$ relate to spherical coordinates?

$$L_k = \sum_{ij} \varepsilon_{kij}\, r_i\, p_j$$

$$(x, y, z) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$L_i = \mathrm{i}\,\varepsilon_{ijk}\, x_j\, \frac{\partial}{\partial x_k} = \mathrm{i}\,\varepsilon_{ijk}\, x_j \left(\frac{\partial y_l}{\partial x_k}\right) \frac{\partial}{\partial y_l} = \mathrm{i}\,\varepsilon_{ijk}\, x_j \left(\frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)}\right)^{-1}_{lk} \frac{\partial}{\partial y_l}$$

$$\sqrt{\frac{4\pi}{3}}\, r\, Y_1^0(\theta, \varphi) = z$$

$$\sqrt{\frac{4\pi}{3}}\, r \left(Y_1^1 + Y_1^{-1}\right) \frac{1}{\sqrt{2}} = x$$

$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$. $\{|j_1, j_2, m_1, m_2\rangle\} =$ basis for Hilbert space.

$$\mathcal{D}(R_\Omega(\theta)) \quad \Big[|\alpha_1\rangle \otimes |\alpha_2\rangle\Big] = (\mathcal{D}_1(R)\,|\alpha_1\rangle) \otimes (\mathcal{D}_2(R)\,|\alpha_2\rangle) = (\mathcal{D}_1(R) \otimes \mathcal{D}_2(R)\,)(|\alpha_1\rangle \otimes |\alpha_2\rangle)$$

$(2 \times 2) \otimes (3 \times 3) = (6 \times 6)$.

$$\mathcal{D}(\delta\theta) = \exp\!\left(-\mathrm{i}\,\boldsymbol{J}_1 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}\right) \otimes \exp\!\left(-\mathrm{i}\,\boldsymbol{J}_2 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}\right) = \left(\mathbb{I} - \mathrm{i}\,\boldsymbol{J}_1 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}\right) \otimes \exp\!\left(\mathbb{I} - \mathrm{i}\,\boldsymbol{J}_2 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}\right) =$$

$$= \mathbb{I} \otimes \mathbb{I} - \mathrm{i}\,\boldsymbol{J}_1 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar} \otimes \mathbb{I} - \mathbb{I} \otimes \mathrm{i}\,\boldsymbol{J}_2 \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar} + \cdots =$$

$$= \mathbb{I}_{6\times 6} - \mathrm{i}\,(\boldsymbol{J}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{J}_2) \cdot \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}$$

$$= \mathbb{I} - \mathrm{i}\,\boldsymbol{J}_{\mathrm{tot}} \otimes \boldsymbol{\Omega}\, \frac{\delta\theta}{\hbar}$$

$$\boldsymbol{J}_{\mathrm{tot}} = \boldsymbol{J}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{J}_2 = \left[\text{this is sloppily written as}\right] = \boldsymbol{J}_1 + \boldsymbol{J}_2$$