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$$[I_i, I_j] = \mathrm{i}\,\varepsilon_{ijk} I_k$$

$$\langle R_\Omega(\theta)\,x|\psi\rangle = \langle x|\mathcal{D}(R_\Omega(\theta))|\psi\rangle$$

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_1 R_2)\text{: group property}$$

$$\mathcal{D}(R^{-1}) = D^\dagger(R)\text{: unitary}$$

$$R_\Omega(\theta) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{I}\theta}$$

$$\mathcal{D}(R_\Omega(\theta)) = \mathrm{e}^{-\mathrm{i}\sum_i \mathcal{D}(I_i)\Omega_i\theta/\hbar} \equiv \mathrm{e}^{\mathrm{i}\boldsymbol{J}\cdot\boldsymbol{\Omega}\theta/\hbar}$$

Lie Algebra for the rotations:

$$[J_i, J_j] = \mathrm{i}\,\hbar\,\varepsilon_{ijk} J_K$$

Two-dimensoinal representation of rotations:

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2\,\mathrm{i}\,\varepsilon_{ijk}\,\sigma_k$$

$$[S_i, S_j] = \mathrm{i}\,\hbar\,\varepsilon_{ijk}\,S_k$$

$$\{\sigma_i, \sigma_j\} = 2\,\delta_{ij}$$

$$\boldsymbol{\alpha} = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

Pauli-matrix algebra

$$\boldsymbol{\alpha}\,\boldsymbol{\beta} = \alpha\cdot\beta + \mathrm{i}\,\boldsymbol{\alpha}\times\boldsymbol{\beta}$$

$$|\boldsymbol{\Omega}| = 1, \quad \boldsymbol{\Omega}^n = \begin{cases} 1, & n\,\text{even} \\ \Omega, & n\,\text{odd} \end{cases}$$

$$\mathcal{D}(R_\Omega(\theta)) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}\theta/2} = 1 + \frac{-\mathrm{i}}{2}\boldsymbol{\Omega}\,\theta - \frac{\theta^2}{4} + \frac{\mathrm{i}}{6}\boldsymbol{\Omega}\theta^3 + \cdots = \cos\frac{\theta}{2} - \mathrm{i}\,\boldsymbol{\Omega}\sin\frac{\theta}{2}$$

$\mathrm{Rot}(3D) \sim \mathrm{SU}(2)$.

Spin-statistics: Fermon $\Longleftrightarrow$ half integer spin. Boson $\Longleftrightarrow$ integer spin.

$$[J_i, J_j] = \mathrm{i}\,\hbar\,\varepsilon_{ijk} J_k$$

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$$\left[J^2, J_z\right] = 0$$

$$J^2|j, m\rangle = \lambda_J|j, m\rangle$$

$$m|j, m\rangle = \hbar m|j, m\rangle$$

$$\langle j, m|\mathcal{D}(R_\Omega(\theta))|j', m'\rangle = \delta^{jj'} d^j_{m,m'}(R_\Omega(\theta))$$

$$\langle j, m|\mathrm{e}^{\mathrm{i}\boldsymbol{J}\cdot\boldsymbol{\Omega}\theta/\hbar}|j', m'\rangle = 0 \text{ if } j \neq j'$$

$$J_+ = J_x + \mathrm{i}\,J_y, \quad J_- = J_x - \mathrm{i}\,J_y, \quad J_-^\dagger = J_+$$

$$[J_+, J_-] = 2\,\hbar\,J_z$$

$$[J_z, J_\pm] = \pm\,\hbar\,J_\pm$$

$$\left[n, a^\dagger\right] = +\,a^\dagger$$

$$[n, a] = -\,a$$

$$J_z J_+|j, m\rangle = (m+1)\hbar\,J_+|j, m\rangle$$

$$J_z J_-|j, m\rangle = (m-1)\hbar J_-|j, m\rangle$$

$$(J_+J_- + J_-J_+) = \frac{1}{2}\left(J_x^2 + J_y^2\right)$$

$$J^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+)$$

$$\langle j, m|J^2 - J_z^2|j, m\rangle = \frac{1}{2}(\langle j, m|J_+J_-|j, m\rangle + \langle j, m|J_-J_+|j, m\rangle) =$$

$$= \frac{1}{2}\left(|J_-|j, m\rangle|^2 + |J_+|j, m\rangle|^2\right)$$

$$\lambda_j^2 \geqslant m^2 \geqslant 0$$

Since $\lambda_j^2 > m^2, \exists m_{\max}\colon J_+ |m_{\max}\rangle = 0,\ J_- |m_{\min}\rangle = 0.$