2008–09–24

3.14 a) $\boldsymbol{J}^2 = J_z^2 + J_+ J_- - \hbar\, J_z$

$$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$$

$$J_\pm = J_x \pm \mathrm{i}\, J_y \quad \Rightarrow \quad J_x = \frac{1}{2}(J_+ + J_-), \quad J_y = \frac{1}{2\,\mathrm{i}}(J_+ - J_-)$$

$$\boldsymbol{J}^2 = \frac{1}{4}\Big\{(J_+ + J_-)^2 - (J_+ - J_-)^2 + 4\,J_z^2\Big\} = \frac{1}{2}\{J_+J_- + [J_-, J_+] + J_+J_-\} + J_z^2 =$$

$$= J_+J_- - \frac{2\hbar\, J_z}{2} + J_z^2 = J_z^2 + J_+J_- - \hbar J_z$$

b) $J_-\psi_{j,m} = c_-\psi_{j,m-1}$.

$$\langle j, m | J_+ J_- | j, m\rangle = |c_-|^2$$

$$J_+J_- = \boldsymbol{J}^2 - J_z^2 + \hbar\, J_z$$

$$\langle j, m | \big(\boldsymbol{J}^2 - J_z^2 + \hbar\, J_z\big) | j, m\rangle =$$

$$= \hbar^2\, j\,(j+1) - (m\,\hbar)^2 + \hbar \cdot \hbar\, m \underbrace{\langle j, m | j, m\rangle}_{=1} = |c_-|^2$$

$$c_- = \hbar\,\sqrt{j\,(j+1) - m^2 + m}$$

3.16 $\langle L_x\rangle, \langle L_y\rangle, \langle L_x^2\rangle, \langle L_y^2\rangle$.

$$\langle L_x\rangle = \langle l, m | \frac{1}{2}(L_+ + L_-) | l, m\rangle =$$

$$= \frac{\hbar}{2}\Big\{\sqrt{(1-m)(l+m+1)}\,\langle l, m | l, m+1\rangle + \sqrt{(l+m)(l-m+1)}\langle l, m | l, m-1\rangle\Big\} = 0$$

$$\langle L_y\rangle = \langle l, m | \frac{1}{2\,\mathrm{i}}(L_+ - L_-) | l, m\rangle = 0$$

$$\langle L_x^2\rangle = \frac{1}{4}\langle l, m | \big(L_+^2 + L_-^2 + L_+L_- + L_-L_+\big) | l, m\rangle = \frac{1}{2}\hbar^2\{(l-m)(l+m+1) + m\}\cdot 1 =$$

$$= \frac{\hbar^2}{2}\big\{l\,(l+1) - m^2\big\}$$

$$\langle L_y^2\rangle = \frac{\hbar^2}{2}\big\{l(l+1) - m^2\big\}$$

$$\langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = \hbar^2\big\{l\,(l+1) - m^2\big\} + m^2\hbar^2 = l\,(l+1)\,\hbar^2 = \langle \boldsymbol{L}^2\rangle$$

3.15 $\langle \boldsymbol{x} | l, m\rangle$

$$\psi(\boldsymbol{x}) = (x + y + 3z)\, f(r)$$

$$\boldsymbol{L}^2$$

$$\psi(\boldsymbol{x}) = r\,(\sin\theta\cos\varphi + \sin\theta\sin\varphi + 3\cos\theta)\, f(r)$$

$$Y_{1,1} = A\sin\theta\,\mathrm{e}^{\mathrm{i}\varphi}; \quad Y_{1,-1} = -\,A\sin\theta\,\mathrm{e}^{-\mathrm{i}\varphi}; \quad Y_{1,0} = \sqrt{2}\,A\cos\theta; \quad A = \sqrt{\frac{3}{8\pi}}$$

$$\psi(\boldsymbol{x}) = r\,f(r)\bigg\{\frac{3}{\sqrt{2}}\,Y_{1,0} + \frac{1-\mathrm{i}}{2}\,Y_{1,1} - \frac{1+\mathrm{i}}{2}\,Y_{1,-1}\bigg\}\frac{1}{A}$$

All these spherical harmonics have $l=1$.

b. Measure L_z.

Outcomes: $m=0,\pm 1$.

$$\psi(\boldsymbol{x})=r\,f(r)\,F(\theta,\varphi)$$

$$F(\theta,\varphi)=\sqrt{\frac{4\pi}{3}}\left\{3\,Y_{1,0}+\frac{1-\mathrm{i}}{\sqrt{2}}Y_{1,1}-\frac{1-\mathrm{i}}{\sqrt{2}}Y_{1,-1}\right\}$$

$$c_0=3,\quad c_1=\frac{1-\mathrm{i}}{\sqrt{2}},\quad c_{-1}=-\frac{1-\mathrm{i}}{\sqrt{2}}$$

$$P(m=-1)=\frac{|c_{-1}|^2}{|c_0|^2+|c_1|^2+|c_{-1}|^2}=\frac{1}{11}=P(m=1)$$

$$P(m=0)=1-\frac{1}{11}-\frac{1}{11}=\frac{9}{11}$$

c.

$$H=\frac{p^2}{2m}+V$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi+V(r)\,\psi=E\psi$$

$$\psi(\boldsymbol{x})=R(r)\,F(\theta,\varphi)$$

$$\nabla^2\psi=\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{l(l+1)R(r)}{r^2}\right\}F(\theta,\varphi)$$

$l=1$:

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2\,R(r)}{r^2}\right\}+V(r)\,R=E\,R$$

$$V(r)=E-\frac{\hbar^2}{m\,r^2}+\frac{\hbar^2}{2m}\frac{1}{r\,R}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\,R)$$

3.17

$$L_+Y_{\frac{1}{2},\frac{1}{2}}(\theta,\varphi)=0$$

$$Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\varphi)$$

$$Y_{\frac{1}{2},\frac{1}{2}}(\theta,\varphi)\propto\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\sin\theta}$$

a.

$$\frac{1}{\hbar}L_- Y_{\frac{1}{2},\frac{1}{2}} \propto Y_{\frac{1}{2},-\frac{1}{2}} \propto \mathrm{e}^{-\mathrm{i}\varphi}\Big(-\frac{\partial}{\partial\theta}+\mathrm{i}\cot\theta\,\frac{\partial}{\partial\varphi}\Big)\Big(\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\sin\theta}\,\Big)=$$

$$=\ldots=-\,\mathrm{e}^{-\mathrm{i}\varphi/2}\cot\theta\,\sqrt{\sin\theta}$$

b.

$$L_- Y_{\frac{1}{2},-\frac{1}{2}}=0, \qquad \Rightarrow Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\varphi)\propto \mathrm{e}^{-\mathrm{i}\varphi/2}\sqrt{\sin\theta}$$

Contradiction $\Rightarrow l$ half integers not allowed.

3.18

$$\mathcal{D}(R)|l,m\rangle=\sum_{m'}|l,m'\rangle\langle l,m'|\mathcal{D}(R)|l,m\rangle=\sum_{m'}|l,m'\rangle\mathcal{D}^{(L)}_{m',m}(R)$$

$$|l=2,m=0\rangle$$

$$\mathcal{D}^{(2)}_{m',0}(R)$$

Rotation about y axis with angle β.

$$|\mathcal{D}^{(2)}_{m',0}(\alpha=0,\gamma,\gamma=0)|^2=|\sqrt{\frac{4\pi}{2l+1}}\,Y_2^{m'*}(\theta=\beta,\varphi=0)|^2$$

$m=0$:

$$P(m'=0)=|\mathcal{D}^{(2)}_{0,0}(0,\beta,0)|^2=\frac{1}{4}\big(3\cos^2\beta-1\big)^2$$

$m'=\pm1$:

$$P(m'=\pm1)=\frac{3}{2}\cos^2\beta\sin^2\beta$$

$$P(m'=\pm2)=\frac{3}{8}\sin^4\beta$$

$$P(0)+P(1)+P(-1)+P(2)+P(-2)=1$$

3.22 $j=1$

$$\langle j=1,m'|J_y|j=1,m\rangle$$

$$\langle j',m'|J_\pm|j,m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,\delta_{j'j}\delta_{m',m\pm1}$$

$$J_\pm=\begin{pmatrix}\square & & \\ & \square & \\ & & \square\end{pmatrix}$$

Blockdiagonal.

$$J_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{2}\,\mathrm{i} & 0 \\ \sqrt{2}\,\mathrm{i} & 0 & -\sqrt{2}\,\mathrm{i} \\ 0 & \sqrt{2}\,\mathrm{i} & 0 \end{pmatrix}$$

b.

$$\exp\left(\frac{-\mathrm{i}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\varphi}{2}\right) = I\cos\frac{\varphi}{2} - \mathrm{i}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\frac{\varphi}{2}$$

$$\exp\left(-\frac{\mathrm{i}\,J_y\,\beta}{\hbar}\right)$$

$$\left(\frac{J_y}{\hbar}\right)^{2m} = \left(\frac{J_y}{\hbar}\right)^2, \quad \left(\frac{J_y}{\hbar}\right)^{2m+1} = \frac{J_y}{\hbar}$$

$$\exp\left(-\frac{\mathrm{i}\,J_y\,\beta}{\hbar}\right) = I - \frac{\mathrm{i}\,J_y\,\beta}{\hbar} + \frac{(-\mathrm{i})^2\,\beta^2}{2!}\left(\frac{J_y}{\hbar}\right)^2 + \cdots =$$

$$= I - \mathrm{i}\,\frac{J_y}{\hbar}\underbrace{\left(\beta - \frac{\beta^3}{3!} + \cdots\right)}_{=\sin\beta} + \left(\frac{J_y}{\eta}\right)^2\underbrace{\left(-\frac{\beta^2}{2!} + \cdots\right)}_{\cos\beta - 1} =$$

$$= I - \mathrm{i}\,\frac{J_y}{\hbar}\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\beta)$$

$j = 1$:

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

$$\mathcal{D}^{(j)}(\alpha,\beta,\gamma) = \exp\left(-\frac{\mathrm{i}\,J_z\,\alpha}{\hbar}\right)\exp\left(\frac{-\mathrm{i}\,J_y\,\beta}{\hbar}\right)\exp\left(\frac{-\mathrm{i}\,J_z\,\gamma}{\hbar}\right)$$

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3.8

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma)=\exp\left(-\frac{\mathrm{i}\,\sigma_3\,\alpha}{2}\right)\exp\left(\frac{-\mathrm{i}\,\sigma_2\,\beta}{2}\right)\exp\left(\frac{-\mathrm{i}\sigma_3\,\gamma}{2}\right)=$$

$$=\begin{pmatrix} \exp\left(\frac{-\mathrm{i}(\alpha+\gamma)}{2}\right)\cos\frac{\beta}{2} & -\exp\left(\frac{-\mathrm{i}\,(\alpha-\gamma)}{2}\right)\sin\frac{\beta}{2} \\ \exp\left(\frac{\mathrm{i}(\alpha-\gamma)}{2}\right)\sin\frac{\beta}{2} & \exp\left(\frac{\mathrm{i}\,(\alpha+\gamma)}{2}\right)\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\begin{pmatrix} \cos\frac{\phi}{2}-\mathrm{i}\,n_z\sin\frac{\phi}{2} & (-\mathrm{i}\,n_x-n_y)\sin\frac{\phi}{2} \\ (-\mathrm{i}\,n_x+n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2}+\mathrm{i}\,n_z\sin\frac{\phi}{2} \end{pmatrix}=\exp\left(-\frac{\mathrm{i}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)$$

$(1,1)+(2,2)\Rightarrow\phi$.

$$\mathcal{D}=\cos\frac{\phi}{2}I-\mathrm{i}\,\sigma_i\,n_i\sin\frac{\phi}{2}$$

$$\mathrm{Tr}(D)=2\cos\frac{\phi}{2}$$

$$\mathrm{Tr}(\sigma_j\mathcal{D})=\mathrm{Tr}(\cos\frac{\phi}{2}\sigma_j-\mathrm{i}\,\sigma_j\,\sigma_i n_i\sin\frac{\phi}{2})=$$

$$=-\mathrm{i}\,\mathrm{Tr}\left(\frac{1}{2}[\sigma_j,\sigma_i]+\frac{1}{2}\{\sigma_j,\sigma_i\}\right)n_i$$

$$=\ldots=-2\ \mathrm{i}\,n_j\sin\frac{\phi}{2}$$