2008–09–23

2.20

$$V = \begin{cases} \frac{1}{2} k\, x^2, & x > 0 \\ +\infty, & x < 0 \end{cases}$$

Calculate $\langle x^2 \rangle$ in the ground state.

For $x > 0$ ψ obeys the simple harmonic oscillator. When $x < 0$, $\psi = 0$.

$\Rightarrow E_n = (2n+1)\frac{\hbar\,\omega}{2}$, $n = 1, 3, 5, ...$ We can rename this as $E_k = (4\,k+3)\,\hbar\,\omega/2$, $k = 0, 1, 2, ...$

$$E_0 = \frac{3\,\hbar\,\omega}{2}$$

$$\langle x^2 \rangle =_{\text{SHO}} \langle 1|x^2|1\rangle_{\text{SHO}} = [\text{Problem 2.13}] = \frac{\hbar}{2m\omega}\Big\{\sqrt{1\cdot 1}\;\delta_{0,0} + \sqrt{2\cdot 2}\;\delta_{2,2}\Big\} = \frac{3\,\hbar}{2\,m\,\omega}$$

2.28

$$K(x'', t; x', t_0) = \langle x''|\exp\!\left(-\frac{\mathrm{i}\,H\,(t-t_0)}{\hbar}\right)|x'\rangle$$

$$H = \frac{p^2}{2m}$$

$$K(x'', t; x', t_0) = \sum_{a} \langle x''|a'\rangle\,\langle a'|x'\rangle\,\exp\!\left(-\frac{\mathrm{i}\,E_{a'}(t-t_0)}{\hbar}\right) =$$

$$= \frac{1}{2\,\pi\,\hbar}\int_{-\infty}^{\infty} \mathrm{d}p'\,\exp\!\left\{-\frac{\mathrm{i}\,p'^{\,2}}{2m}\,\frac{(t-t_0)}{\hbar} + \frac{\mathrm{i}\,p''(\,x''-x')}{\hbar}\right\} =$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p'\,\exp\!\left\{-\frac{\mathrm{i}\,(t-t_0)}{2\,m\,\hbar}(p'^{\,2} - \frac{p'(x''-x')/2m}{t-t_0}\right\} =$$

$$= \frac{1}{2\pi\hbar}\exp\!\left\{\frac{\mathrm{i}\,m\,(x'-x'')^2}{2\hbar(t-t_0)}\right\}\sqrt{\frac{\pi\,2\,m\hbar}{\mathrm{i}\,(t-t_0)}}$$

2.19

$$\begin{aligned} J_{\pm} &= \hbar\,a_{\pm}^{\dagger}\,a_{\mp} \\ J_z &= \frac{\hbar}{2}\Big(a_+^{\dagger}a_+ - a_-^{\dagger}a_-\Big) = \frac{\hbar}{2}(N_+ - N_-) \end{aligned}$$

$$\Big[a_{\pm}^{\dagger}, a_{\pm}\Big] = -1; \quad [a_+, a_-] = \Big[a_+^{\dagger}, a_-\Big] = 0$$

$$[J_z, J_{\pm}] = \pm\,\hbar\,J_{\pm}, \quad [\boldsymbol{J}^2, J_z] = 0$$

$$[J_z, J_+] = \frac{\hbar^2}{2}\Big[a_+^{\dagger}a_+ - a_-^{\dagger}a_-, a_+^{\dagger}a_-\Big] = \frac{\hbar^2}{2}\Big\{a_+^{\dagger}\Big[a_+, a_+^{\dagger}\Big]a_- - a_+^{\dagger}\Big[a_-^{\dagger}, a_-\Big]a_-\Big\} = \hbar^2 a_+^{\dagger}a_- = \hbar\,J_+$$

$$[J_z, J_-] = -\,\hbar J_-$$

$$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2 = J_+J_- - \hbar\,J_z + J_z^2$$

$$J_{\pm} = J_x \pm i\,J_y$$

$$\big[\boldsymbol{J}^2, J_z\big] = \big[J_+J_- - \hbar\,J_z + J_z^2, J_z\big] = [J_+, J_z]J_- + J_+[J_-, J_z] = -\,\hbar J_+J_- + J_+\hbar J_- = 0$$

$$\boldsymbol{J}^2 = J_+J_- - \hbar J_z + J_z^2 = ... = \hbar^2 N\left(\frac{N}{2} + 1\right)$$

3.2

$$U=\frac{a_0+\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}}{a_0-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}}$$

$a_0, \boldsymbol{a}$ are real parameters.

a) U is unitary ($U\,U^{\dagger}=U^{\dagger}U=1$), unimodular ($\det U=1$). 2×2 matrix. These are the SU(2) matrices.

$$\boldsymbol{A}=a_0+\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}=\begin{pmatrix} a_0+\mathrm{i}\,a_3 & a_2+\mathrm{i}\,a_1\\ -a_2+\mathrm{i}\,a_1 & a_3-\mathrm{i}\,a_3\end{pmatrix}$$

$$\boldsymbol{A}^{\dagger}=a_0-\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}$$

$$\boldsymbol{A}^{\dagger}\boldsymbol{A}=\boldsymbol{A}\,\boldsymbol{A}^{\dagger}=a_0+(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2=\left\{\sigma_i a_i\,\sigma_j a_j=a_i a_j\left\{\frac{1}{2}[\sigma_i,\sigma_j]+\{\sigma_i,\sigma_j\}\right\}=a_i a_i I=|\boldsymbol{a}|\,I\right\}=$$

$$\{\quad[\sigma_i,\sigma_j]=2\,\mathrm{i}\,\varepsilon_{ijk}\sigma_k,\quad\{\sigma_i,\sigma_j\}=2\,\delta_{ij}I\quad\}$$

$$=a_0^2+|\boldsymbol{a}|^2=|a|^2\,I$$

$$\left(A^{\dagger}\right)^{-1}=\frac{1}{|a|^2}\,A$$

$$U=A\left(A^{\dagger}\right)^{-1}=\frac{1}{|a|^2}\,A^2$$

$$U^{\dagger}U=\left(\frac{1}{|a|^2}\right)^2A^{\dagger}\,A^{\dagger}\,A\,A=I$$

This shows unitarity.

$$\det U=\det\left(\frac{A^2}{|a|^2}\right)=\left(\det\left(\frac{A}{|a|}\right)\right)^2=\left(\frac{1}{|a|^2}\det A\right)^2=1$$

$$\mathcal{D}^{(1/2)}(\boldsymbol{n},\psi)=\exp\left(-\frac{\mathrm{i}}{2}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\,\psi\right)=\cos\frac{\psi}{2}-\mathrm{i}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\sin\frac{\psi}{2}$$

$$U=\frac{A^2}{|a|^2}=\frac{1}{|a|^2}(a_0+\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a})^2=\frac{1}{|a|^2}\left(a_0+2\,\mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}-(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2\right)=$$

$$=\frac{1}{|a|^2}\left(a_0^2-|\boldsymbol{a}|^2\right)+\mathrm{i}\,\frac{2\,a_0}{|a|^2}\,\boldsymbol{\sigma}\cdot\boldsymbol{a}=\tilde{a}_0^2-|\tilde{\boldsymbol{a}}|^2+\mathrm{i}\,2\,\tilde{a}_0\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{a}}$$

$$|\tilde{a}|^2=\tilde{a}_0^2+|\tilde{\boldsymbol{a}}|^2=1$$

$$\cos\frac{\psi}{2}=\tilde{a}_0^2-|\tilde{\boldsymbol{a}}|^2$$

$$-\,n_i\sin\frac{\psi}{2}=2\tilde{a}_0\tilde{a}_i$$

$$\sin\frac{\psi}{2}=\pm\sqrt{1-\cos^2\frac{\psi}{2}}=\pm\,2\,|\tilde{a}_0|\,|\tilde{\boldsymbol{a}}|$$

$$\cos\frac{\psi}{2}=1-2\,|\tilde{\boldsymbol{a}}|^2$$

$$n_i=\mp\,\frac{\tilde{a}_0\,\tilde{a}_i}{|\tilde{a}_0||\tilde{a}|}$$

3.3

$$H = A\,\boldsymbol{S}^{(e^-)}\cdot\boldsymbol{S}^{(e^+)} + \frac{e\,B}{m\,c}\Big(S_z^{(e^-)} - S_z^{(e^+)}\Big)$$

$$\chi_+^{(e^-)}\chi_-^{(e^+)} \qquad \begin{pmatrix}1\\0\end{pmatrix}_{e^-}\otimes\begin{pmatrix}0\\1\end{pmatrix}_{e^+}$$

a. $A\to 0$

$$H\chi_+^{(-)}\chi_-^{(+)} = \frac{e\,B}{m\,c}\Big\{\Big(S_z^{(e^-)}\chi_+^{(-)}\Big)\chi_-^{(+)} - \chi_+^{-}\Big(S_z^{(+)}\chi_-^{(+)}\Big)\Big\} = \frac{e\,B}{m\,c}\Big(\frac{\hbar}{2}\chi_+^{(-)}\chi_-^{(+)} - \chi_+^{(-)}\Big(-\frac{\hbar}{2}\Big)\chi_-^{(+)}\Big)$$

$$= \frac{e\,B\,\hbar}{m\,c}\,\chi_+^{(-)}\chi_-^{(+)}$$

b) $A\neq 0, e\,B/m c\to 0.$

$$\frac{1}{A}\,H = \boldsymbol{S}^{(-)}\cdot\boldsymbol{S}^{(+)} = S_x^{(-)}S_x^{(+)} + S_y^{(-)}S_y^{(+)} + S_z^{(-)}S_z^{(+)}$$

$$S_+|+\rangle = 0, \quad S_+|-\rangle = |+\rangle, \quad S_-|+\rangle = |-\rangle, \quad S_-|-\rangle = 0$$

$$S_x^{\pm} = \frac{1}{2}(S_+^{\pm} + S_-^{\pm}), \quad S_y^{-} = \frac{1}{2\,\mathrm{i}}(S_+^{\pm} - S_-^{\pm})$$

$$\frac{1}{A}H = \frac{1}{2}\Big(S_x^{(-)}S_-^{(+)} + S_-^{(-)}S_+^{(+)}\Big) + S_z^{(-)}S_z^{(+)}$$

$$H\chi_+^{(-)}\chi_-^{(+)} = A\Big\{\frac{\hbar^2}{2}\Big(\chi_-^{(-)}\chi_+^{(+)}\Big) + \frac{\hbar^2}{4}\chi_+^{(-)}\chi_-^{(+)}\Big\}$$

$\chi_+^{(-)}\chi_-^{(+)}$ is not an eigenstate of H.

$$\langle H\rangle = \Big(\chi_-^{(+)}\Big)^\dagger\Big(\chi_+^{(-)}\Big)^\dagger H\chi_+^{(-)}\chi_-^{(+)} = -\frac{\hbar^2}{4}\,A$$

3.4 Spin 1.

$$\{|S=1, S_z=m\rangle; m=+1,0,-1\} = \{|+\rangle, |0\rangle, |-\rangle\}$$

$s, 2s+1.$

$$S_z(S_z+1)(S_z-1)$$

$$S_x(S_x+1)(S_x-1)$$

$$S_z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$S_z(S_z+1)(S_z-1) = 0$$

By a symmetry argument, we should get $S_x(S_x+1)(S_x-1)$.

$$\langle j', m'|J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\;\delta_{j'j}\,\delta_{m',m\pm1}$$

$$S_+ \doteq \begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}\begin{matrix}m'=1\\m'=0\\m'=-1\end{matrix}$$

$$S_- = S_+^\dagger$$

$$S_x = \frac{1}{2}(S_+ + S_-)$$