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$$p = \frac{\partial S(q)}{\partial q}$$

$$0 = K(P, Q) = H + \frac{\partial S}{\partial t}$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = -\frac{\partial S}{\partial t}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial q}\,\dot{q} + \frac{\partial S}{\partial t} = p\,\dot{q} - H = L(q, \dot{q})$$

$$S(t) = \int_0^t L(q, \dot{q})\mathrm{d}t$$

The action S is actually the Hamilton-Jacobi function S.

$$\psi(t) = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\Phi(x,t)/\hbar}$$

$$\mathrm{i}\,\hbar\,\frac{\partial}{\partial t}\psi(t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x, t)$$

$$-\frac{\partial \Phi}{\partial t} = \frac{1}{\hbar}\frac{\hbar^2}{2\,m}\left(\frac{\mathrm{i}\partial\Phi}{\partial x}\right)^2\psi(t) + V(x)\psi(x)$$

$$-\frac{\partial \Phi}{\partial t} = \frac{1}{2m}\left(\frac{\partial \Phi}{\partial t}\right)^2 + V(x)$$

The phase sitting in the exponential is the classical action from the Hamilton-Jacobi equation. In the classical limit $\Phi \rightarrow S$.

$$\psi(x) = \mathrm{e}^{\mathrm{i}\Phi/\hbar} = \sum_{x_n} \mathrm{e}^{\mathrm{i}S_0(x)} + \mathrm{e}^{\mathrm{i}S_1(x)} + \mathrm{e}^{\mathrm{i}S_2(x)}$$

$$\langle x(t)|x_0(t)\rangle = \int\!\!\int D x(t)\,\mathrm{e}^{\mathrm{i}/\hbar\int L(x,\dot{x})\mathrm{d}t}$$

$$\langle x(t)|x(0)\rangle = A^N\int\cdots\int \mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{N-1}\exp\left(\frac{\mathrm{i}}{\hbar}\,\varepsilon\sum_{n=0}^{N}\frac{1}{2}m\left(\frac{x_{n+1}-x_n}{\varepsilon}\right)^2 - V(x_n)\right)$$

$$\langle x(t)|x(0)\rangle = \int\cdots\int \langle x_N|x_{N-1}\rangle\langle x_{N-1}|...|x_n\rangle\langle x_n|x_{n-1}\rangle...\langle x_1|x_0\rangle\,\mathrm{d}x_1...\mathrm{d}x_{N-1}$$

$$A\exp\left(\frac{\mathrm{i}\varepsilon}{\hbar}\frac{1}{2}m\left(\frac{x_{n+1}-x_n}{\varepsilon^2}\right) - V(x_n)\right) \stackrel{?}{=} \langle x_{n+1}|\mathrm{e}^{-\mathrm{i}\varepsilon H/\hbar}|x_n\rangle$$

$$\langle x'|\exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]\right)|x\rangle = \int \mathrm{d}p\,\langle x'|\exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}\frac{p^2}{2m}\right)|p\rangle\langle p|\exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar}V(x)\right)|x\rangle =$$

(OK in the limit $\varepsilon \to 0$)

$$= \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} V(x) \right) \int \frac{\mathrm{d}p}{2\pi\hbar} \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} \cdot \frac{p^2}{2\,m} \right) \langle x'|p\rangle \langle p|x\rangle =$$

$$= \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} V(x) \right) \int \frac{\mathrm{d}p}{2\pi\hbar} \exp(\mathrm{i}\, p(x'-x)/\hbar) \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} \frac{p^2}{2m} \right): \text{Gaussian}$$

$$= \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi m\,\hbar}{\mathrm{i}\varepsilon}}\, x \exp\left(\frac{\mathrm{i}\varepsilon}{2\,m\,\hbar} \left(\frac{x-x'}{\varepsilon} \right)^2 - \frac{\mathrm{i}\varepsilon}{\hbar} V(x) \right)$$

Rotations in three dimensions.

a

Invariants $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{ij} a_i\, \delta_{ij}\, b_j$, $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \sum_{ijk} \varepsilon^{ijk}\, a_i\, b_j\, c_k$.

$$\varepsilon^{ijk}: \quad \varepsilon^{123} = 1, \quad \varepsilon^{ijk} = -\,\varepsilon^{jik}, \quad \varepsilon^{ijk} = \varepsilon^{kij}$$

$$R_{\hat{\boldsymbol{z}}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = a\,\delta_{ij} + b\,\hat{\boldsymbol{z}}_i\,\hat{\boldsymbol{z}}_j + c\,\varepsilon_{ijk}\hat{\boldsymbol{z}}_k = \begin{pmatrix} a & c & 0 \\ -\,c & a & 0 \\ 0 & 0 & a+b \end{pmatrix}$$

$$\begin{cases} a = \cos\theta \\ b = 1 - \cos\theta \\ c = -\sin\theta \end{cases}$$

$$R_{\hat{\boldsymbol{\Omega}}}(\theta) = \cos\theta\,\delta_{ij} + (1 - \cos\theta)\hat{\boldsymbol{\Omega}}_i\,\hat{\boldsymbol{\Omega}}_j - \sin\theta\,\varepsilon_{ijk}\,\hat{\boldsymbol{\Omega}}_k$$

$$\left.\frac{\mathrm{d}R_{\hat{\boldsymbol{\Omega}}}}{\mathrm{d}\theta}\right|_{\theta=0} = -\,\varepsilon_{ijk}\Omega_k = \sum_k I_k \hat{\Omega}_k$$

$$I_x = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}; \quad I_y = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad I_z = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[I_x, I_y] = \mathrm{i} I_z$$

$$[I_i, I_j] = \mathrm{i} \sum_k \varepsilon^{ijk} I_k$$

$$R_{\boldsymbol{\Omega}}(\theta) = \mathrm{e}^{-\mathrm{i}(\boldsymbol{I}\cdot\boldsymbol{\Omega})\theta}$$

$$\boldsymbol{I} \cdot \boldsymbol{\Omega} = \mathrm{i}\begin{pmatrix} 0 & -\,\Omega_z & \Omega_y \\ \Omega_z & 0 & \Omega_x \\ -\,\Omega_y & \Omega_x & 0 \end{pmatrix}$$

$$R_{\boldsymbol{\Omega}}(\delta\theta) = 1 - \mathrm{i}\,(\boldsymbol{I} \cdot \boldsymbol{\Omega})\delta\theta + \cdots$$

$$(1 - \mathrm{i}\delta\theta\, I_y)(1 - \mathrm{i}\,\delta\theta I_x) \to R_y R_x = 1 - \mathrm{i}\,\delta\theta(I_x + I_y) + (\delta\theta)^2 I_y\, I_x$$

$$R_x R_y = 1 - \mathrm{i}\delta\theta\,(I_y + I_x) + (\delta\theta)^2\, I_x\, I_y$$

$$R_y R_x - R_x R_y = (\delta\theta)^2 [I_y, I_x] = -\,(\delta\theta)^2\,\mathrm{i}\, I_z$$