2008–09–18

This is a singlet:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\,)$$

Let's try to figure out why.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\,|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\,)$$

Rotationally invariant:

$$R(\theta) = \cos\left(\frac{\theta}{2}\right)?? - \mathrm{i}\,\sigma \cdot \Omega \sin\frac{\theta}{2}$$

$$R_z = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta/2} \end{pmatrix}$$

$$|\psi\rangle \to \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\theta/2}|\uparrow\rangle \otimes \mathrm{e}^{-\mathrm{i}\theta/2}|\downarrow\rangle - \ldots\right) = \frac{1}{\sqrt{2}}(\,|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\,)$$

Rotate about y axis with an angle π.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{array}{c} |\uparrow\rangle \mapsto |\downarrow\rangle \\ |\downarrow\rangle \mapsto -|\uparrow\rangle \end{array}$$

Rotation about x axis:

$$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

2.3 Magnetic field and spin. The spin precesses around the z axis.

$$\begin{pmatrix} \lambda_1^* & \\ & \lambda_2^* \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = \begin{pmatrix} a_{11}\lambda_1^*\lambda_1 & a_{12}\,\lambda_1^*\lambda_2 \\ a_{21}\,\lambda_2^*\lambda_1 & a_{22}\lambda_2^*\lambda_2 \end{pmatrix}$$

$$H = -\,\mu_B\,\boldsymbol{S}\cdot\boldsymbol{B} = -\,g\,\mu\,B\,\frac{\hbar}{2}\sigma_z = -\,\frac{1}{2}\omega\hbar\sigma_z$$

$$S(t) = \begin{pmatrix} \exp\left(-\frac{\mathrm{i}}{2}\,\omega\,t\right) & \\ & \exp\left(\frac{\mathrm{i}}{2}\,\omega\,t\right) \end{pmatrix}\begin{pmatrix} S_z & S_x - i\,S_y \\ S_x + \mathrm{i}\,S_y & -\,S_z \end{pmatrix}\begin{pmatrix} \exp\left(\frac{\mathrm{i}}{2}\,\omega\,t\right) & \\ & \exp\left(-\frac{\mathrm{i}}{2}\,\omega\,t\right) \end{pmatrix}$$

$$= \mathrm{e}^{\mathrm{i}Ht/\hbar}\,\boldsymbol{S}\,\mathrm{e}^{-\mathrm{i}Ht/\hbar}$$

$$S_x + \mathrm{i}\,S_y = S_\perp\,\mathrm{e}^{\mathrm{i}\varphi}, \quad S_\perp = S_x^2 + S_y^2$$

$$S(t) = \begin{pmatrix} S_z & S_\perp \exp(-\,\mathrm{i}\varphi + \mathrm{i}\,\omega t) \\ S_\perp \exp(\mathrm{i}\varphi - \mathrm{i}\,\omega t) & S_z \end{pmatrix}$$

S_z is constant.

$$H = \frac{1}{2}\left(a^\dagger a + \lambda(a^\dagger + a)\right)$$

$$H = \frac{1}{2}(p^2 + x^2) + \tilde{\lambda}\,x$$

Make a canonical transformation:

$$a^\dagger \to a^\dagger - 2\lambda, \quad a \to a - 2\lambda$$

$$\left[a, a^\dagger\right] \to [$$

$$H = \frac{1}{2}\left(a^\dagger - 2\lambda\right)(a - 2\lambda) + \lambda\left(a^\dagger + a\right) = \frac{1}{2}\left(a^\dagger a + \text{illegible}\right) = \frac{1}{2}a^\dagger a + \frac{1}{2}\lambda^2$$

$$a^\dagger \to \gamma^* a^\dagger + \lambda$$
$$a \to \frac{1}{\gamma}a + \lambda^*$$

2.17

$$\langle 0|\mathrm{e}^{\mathrm{i}kx}|0\rangle = \exp\left(-\frac{1}{2}k^2\,\langle 0|x^2|0\rangle\right)$$

$$\langle 0|\exp\left(\mathrm{i}\,\lambda\left(a + a^\dagger\right)\right)|0\rangle = \langle 0|1 + \mathrm{i}\,\lambda\left(a + a^\dagger\right) - \frac{\lambda^2}{2}\left(a + a^\dagger\right)^2 + \cdots|0\rangle =$$

$$= \langle 0|1 - \frac{\lambda^2}{2}\left(\cancel{a}^2 + a\,a^\dagger + \cancel{a^\dagger}a + \left(\cancel{a^\dagger}\right)^2\right) + \cdots|0\rangle$$

"I got stuck"

$$\mathrm{e}^{\mathrm{i}\lambda a^\dagger}\mathrm{e}^{\mathrm{i}\lambda a} = \mathrm{e}^{\mathrm{i}\lambda\left(a + a^\dagger\right) - \frac{1}{2}\left[a^\dagger, a\right] - \frac{1}{12}\left[a^\dagger, \left[a^\dagger, a\right] + \ldots\right]} = \mathrm{e}^{\mathrm{i}\lambda\left(a + a^\dagger\right) - \frac{1}{2}\lambda^2}$$

$$\mathrm{e}^{\mathrm{i}\lambda\left(a + a^\dagger\right)} = \left(\mathrm{e}^{\mathrm{i}\lambda a^\dagger}\mathrm{e}^{\mathrm{i}\lambda a}\right)\mathrm{e}^{-\frac{1}{2}\lambda^2}$$

$$\langle 0|\mathrm{e}^{\mathrm{i}\lambda\left(a + a^\dagger\right)}|0\rangle = \mathrm{e}^{-\frac{1}{2}\lambda^2}\langle 0|\mathrm{e}^{\mathrm{i}\lambda a^\dagger}\mathrm{e}^{\mathrm{i}\lambda a}|0\rangle = \mathrm{e}^{-\frac{1}{2}\lambda^2}$$

Solve problem with canonical transformation.

$$L(q, \dot{q}) = \frac{1}{2q^4}\left(\dot{q}^2 - q^2\right)$$

The differential equation is nasty:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{q}}{q^4} + \frac{\partial}{\partial q}\frac{1}{2}q^{-2}$$

$$\frac{\ddot{q}}{q^4} - \frac{4\dot{q}^2}{q^5} - \frac{1}{q^3} = 0$$

I. Start with the Lagrangian. II. Compute p. III. Construct $H(p, q)$. IV. Canonical P, Q.

Compute p:

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\dot{q}}{q^4} \quad \Rightarrow \quad \dot{q} = p\,q^4$$

Construct $H(p, q)$:

$$H(p, q) = p\dot{q} - L = p^2 q^4 - \frac{1}{2q^4}\left(p^2\,q^8 - q^2\right) = \frac{1}{2}\left(p^2 q^4 + \frac{1}{q^2}\right)$$

Want $Q = \frac{1}{q}$. Need generating function:

$$F(q,P): \quad Q = \frac{\partial F}{\partial P}, \quad p = \frac{\partial F}{\partial q}$$

$$\frac{1}{q} = \frac{\partial F(P,q)}{\partial P}$$

$$F = \frac{P}{q} + f(q,t)$$

$$p = \frac{\partial F}{\partial q} = -\frac{P}{q^2} + \frac{\partial f}{\partial q}$$

Try $f = 0$.

$$q \to \frac{1}{Q}, \quad p \to -\frac{P}{q^2} = -PQ^2$$

$$p = -PQ^2, \quad q = \frac{1}{Q}$$

$$K(P,Q) = H(p,q) - \frac{\partial F}{\partial t}$$

$$K = \frac{1}{2}\left((PQ^2)^2 \frac{1}{Q^2} + Q^2\right) = \frac{1}{2}(P^2 + Q^2)$$

$$Q(t) = A\,\sin t$$

$$P(t) = A \cos t$$

$$q(t) = \frac{1}{Q(t)} = \frac{1}{A \sin t}$$

$$\dot{q}(t) = -\frac{\cos t}{\sin^2 t}$$