2008–09–17

1.33 a)

i. $\langle p'|x|\alpha\rangle = \mathrm{i}\,\hbar\,\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

ii.

$$\langle\beta|x|\alpha\rangle = \int \mathrm{d}p'\,\phi_\beta^*(p')\,\mathrm{i}\hbar\,\frac{\partial}{\partial p'}\,\phi_\alpha(p')$$

$$\phi_\alpha(p') = \langle p'|\alpha\rangle, \quad \phi_\beta(p') = \langle p'|\beta\rangle$$

i.

$$\langle p'|x|p''\rangle = \int \mathrm{d}x'\,\langle p'|x|x'\rangle\,\langle x'|p''\rangle$$

$$\langle x'|p''\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,\mathrm{e}^{\mathrm{i}p''x'/\hbar}$$

$$\langle p'|x|p''\rangle = \int \mathrm{d}x\,x'\,\mathrm{e}^{-\mathrm{i}x'(p'-p'')/\hbar}\frac{1}{2\pi\hbar}$$

$$\frac{\partial}{\partial p'}(\delta(p'-p'')) = \frac{\partial}{\partial p'}\int \mathrm{d}x'\,\mathrm{e}^{-\mathrm{i}x'(p'-p'')/\hbar}\frac{1}{2\pi\hbar} = \frac{1}{2\pi\hbar}\int \mathrm{d}x'\,\frac{x'}{\mathrm{i}\hbar}\mathrm{e}^{\cdots}$$

$$\langle p'|x|p''\rangle = \mathrm{i}\hbar\,\frac{\partial}{\partial p'}\delta(p'-p'')$$

$$\langle p'|x|\alpha\rangle = \int \mathrm{d}p''\,\langle p'|x|p''\rangle\langle p''|\alpha\rangle$$

$$= \int \mathrm{d}p''\left(\mathrm{i}\hbar\,\frac{\partial}{\partial p'}\delta(p-p'')\right)\phi_\alpha(p'')$$

$$= \frac{\partial}{\partial p'}\left\{\int \mathrm{d}p''\,\delta(p'-p'')\phi_\alpha(p'')\,\mathrm{i}\hbar\right\} = \mathrm{i}\hbar\,\frac{\partial}{\partial p'}\phi_\alpha(p')$$

ii.

$$\langle\beta|x|\alpha\rangle = \int \mathrm{d}p'\,\langle\beta|p'\rangle\,\langle p'|x|a\rangle = \int \mathrm{d}p'\,\phi_\beta^*(p')\,\mathrm{i}\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

b) x är en operator: $\exp(\mathrm{i}\,x\,\Xi/\hbar)$.

$$|p',\Xi\rangle = \exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)|p'\rangle$$

$$p|p',\Xi\rangle = p\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)|p'\rangle = \exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)p'|p'\rangle + \left[p,\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)\right]|p'\rangle =$$

$$= (p'+\Xi)|p',\Xi\rangle$$

ty

$$\left[p,\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)\right] = -\,\mathrm{i}\hbar\,\frac{\partial}{\partial x}\left(\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)\right) = \Xi\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)$$

$\exp\!\left(\frac{\mathrm{i}\,x\,\Xi}{\hbar}\right)$ is a momentum tranlation.

2.8

$$H = \delta(|\alpha'\rangle\langle a''| + |a''\rangle\langle a'|), \quad a' \neq a'', \quad \delta \in \mathbb{R}$$

$$H \doteq \delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a) $H|\psi\rangle = E\,|\psi\rangle$.

$$\det(H - E\,I) = 0$$

$$E \in \{+\delta, -\delta\}$$

$$|+\delta\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\delta\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) Antar: vid $t = 0$ så befinner sig systemet i $|\alpha'\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$|\psi(t)\rangle \doteq \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

Schrödinger-ekvationen:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

$$\begin{cases} \mathrm{i}\,\hbar \frac{\partial}{\partial t}A(t) &= \delta B(t) \\ \mathrm{i}\,\hbar \frac{\partial}{\partial t}B(t) &= \delta A(t) \end{cases}$$

$$\begin{cases} \frac{\partial^2}{\partial t^2}A(t) + \left(\frac{\delta}{\hbar}\right)^2 A(t) &= 0 \\ \frac{\partial^2}{\partial t^2}B(t) + \left(\frac{\delta}{\hbar}\right)^2 B(t) &= 0 \end{cases}$$

$$\begin{cases} A(t) = A_1\cos\omega t + A_2\sin\omega t \\ B(t) = B_1\cos\omega t + B_2\sin\omega t \end{cases}; \quad \omega = \frac{\delta}{\hbar}, \quad \begin{cases} A(0) = 1 \\ B(0) = 0 \end{cases}$$

$$\langle\psi(t)|\psi(t)\rangle = 1$$

$$|\psi(t)\rangle = \begin{pmatrix} \cos\omega t \\ -\mathrm{i}\sin\omega t \end{pmatrix}$$

b) Sannolikheten att mäta $|a''\rangle$ blir

$$|\langle a''|\psi(t)\rangle|^2 = |\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\omega t \\ -\mathrm{i}\sin\omega t \end{pmatrix}|^2 = \sin^2\omega t$$

2.2 2-tillståndssystem

$$H \doteq \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix}$$

H är inte hermitesk.

$$\Rightarrow U(t) = \exp\left(-\frac{\mathrm{i}\,H}{\hbar}\right)$$

icke-unitär. Sannolikheten under tidsutveckling kommer inte vara bevarad.

Antag $H_{11} = H_{22} = 0$

$$\Rightarrow H \doteq \begin{pmatrix} 0 & H_{12} \\ 0 & 0 \end{pmatrix}$$

$$|\alpha(0)\rangle = c_{10}|1\rangle + c_{20}|2\rangle \doteq \begin{pmatrix} c_{10} \\ c_{20} \end{pmatrix}$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = H\,|\alpha(t)\rangle$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} 0 & H_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$\begin{cases} \mathrm{i}\,\hbar\,\frac{\partial}{\partial t}c_1(t) = H_{12}\,c_{2(t)} \\ \mathrm{i}\hbar\,\frac{\partial}{\partial t}c_2(t) = 0 \end{cases} \quad \Rightarrow \quad c_2(t) = c_{20}$$

$$\begin{cases} c_1(t) &= -\frac{\mathrm{i}H_{12}}{\hbar}\,c_{20}\,\hbar + c_{10} \\ c_2(t) &= c_{20} \end{cases}$$

$$\langle\alpha(t)|\alpha(t)\rangle = |c_1(t)|^2 + |c_2(t)|^2$$

$$|c_1(t)|^2 \rightarrow \infty, \quad \text{då } t \rightarrow \infty.$$

Sannolikheten är inte bevarad.

2.1

$$H = -\frac{e\,B}{m\,c}S_z = \omega\,S_z$$

$S_x(t), S_y(t), S_z(t)$.

Heisenberg equations of motion:

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \frac{1}{\mathrm{i}\,\hbar}\,[S_i, H]$$

$$[S_i(t), S_j(t)] = \mathrm{i}\,\varepsilon_{ijk}\,S_k(t)$$

$$\frac{\mathrm{d}S_z}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[S_z, \omega S_z] = 0 \quad \Rightarrow S_z(t) = S_z(0)$$

$$\frac{\mathrm{d}S_x}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\,[S_x, \omega S_z] = -\,\omega\,S_y$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[S_y, \omega S_z] = \omega\,S_x$$

$$\frac{\mathrm{d}S_x}{\mathrm{d}t} + \mathrm{i}\,\frac{\mathrm{d}S_y}{\mathrm{d}t} = -\,\omega\,S_y + \mathrm{i}\,\omega S_x = \mathrm{i}\omega\,(S_x + \mathrm{i}\,S_y)$$

$$\frac{\mathrm{d}S_x}{\mathrm{d}t} - \mathrm{i}\,\frac{\mathrm{d}S_y}{\mathrm{d}t} = -\,\mathrm{i}\,\omega(S_x - \mathrm{i}\,S_y)$$

$$S_x(t) + \mathrm{i}\,S_y(t) = (S_x(0) + \mathrm{i}\,S_y(t))\,\mathrm{e}^{\mathrm{i}\omega t}$$

$$S_x(t) - \mathrm{i}S_y(t) = (S_x(0) - \mathrm{i}\,S_y(0))\,\mathrm{e}^{\mathrm{i}\omega t}$$

$$\begin{cases} S_x(t) = S_x(0)\cos\omega t - S_y(0)\sin\omega t \\ S_y(t) = S_x(0)\sin\omega t - S_y(0)\cos\omega t \\ S_z(t) = S_z(0) \end{cases}$$

2.12 $t=0$

$$|\alpha(0)\rangle = \exp\left(-\frac{\mathrm{i}\,p\,a}{\hbar}\right)|0\rangle$$

$H = N + \frac{1}{2}$. a is a parameter, p is the momentum operator.

a)

$$\langle x'|0\rangle = \pi^{-1/4}\,x_0^{-1/2}\exp\left(-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right)$$

$$x_0 = \sqrt{\frac{\hbar}{m\,\omega}}$$

$$\langle x'|\alpha(0)\rangle = \langle x'|\exp\left(-\frac{\mathrm{i}\,p\,a}{\hbar}\right)|0\rangle$$

$$x\exp\left(\frac{\mathrm{i}\,p\,a}{\hbar}\right)|x'\rangle = \left(\left[x,\exp\left(\frac{\mathrm{i}\,p\,a}{\hbar}\right)\right] + \exp\left(\frac{\mathrm{i}\,p\,a}{\hbar}\right)x\right)|x'\rangle =$$

$$= \left(\left[x,\frac{\mathrm{i}\,p\,a}{\hbar}\right] + x\right)\exp\left(\frac{\mathrm{i}\,p\,a}{\hbar}\right)|x'\rangle = (x'-a)\exp\left(\frac{\mathrm{i}\,p\,a}{\hbar}\right)|x'\rangle = |x'-a\rangle$$

$$\langle x'|\alpha(0)\rangle = \langle x'-a|0\rangle$$

b) $\langle 0|\alpha(0)\rangle =$

$$= \int_{-\infty}^{\infty}\mathrm{d}x'\,\langle 0|x'\rangle\,\langle x'|\alpha(0)\rangle =$$

$$= \int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\,\langle 0|\tilde{x}+\frac{a}{2}\rangle\,\langle\tilde{x}-\frac{a}{2}|0\rangle$$

$$= \frac{1}{x_0\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\,\exp\left\{-\frac{\left(\tilde{x}+\frac{a}{2}\right)^2 + \left(\tilde{x}-\frac{a}{2}\right)^2}{2\,x_0^2}\right\} =$$

$$= \frac{1}{x_0\sqrt{\pi}}\exp\left(-\left(\frac{a}{2\,x_0}\right)^2\right)\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\,\exp\left(-\frac{\tilde{x}^2}{x_0^2}\right) = \exp\left(-\left(\frac{a}{2\,x_0}\right)^2\right)$$

$$|c_0|^2 = \exp\left(-\frac{a^2}{2\,x_0^2}\right)$$

At a later time t:

$$c_0(t) = \langle 0|\alpha(0)\rangle = \langle 0|\exp\left(-\frac{\mathrm{i}\,H\,t}{\hbar}\right)|\alpha(0)\rangle =$$

$$= \exp\left(-\frac{\mathrm{i}\,E_0\,t}{\hbar}\right)\langle 0|\alpha(0)\rangle = \exp\left(-\frac{\mathrm{i}\,\omega\,t}{2}\right)c_0$$

$$|c_0(t)|^2 = |c_0|^2$$

2.13 One-dimensional simple harmonic oscillator.

a) Compute $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|...|n\rangle$.

$$x=\sqrt{\frac{\hbar}{2\,m\,\omega}}\,(a+a^{\dagger}\,)$$

$$p=\mathrm{i}\,\sqrt{\frac{\hbar\,m\,\omega}{2}}(a^{\dagger}-a)$$

$$\left(\begin{array}{c} a \\ a^{\dagger} \end{array}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(x\pm\frac{\mathrm{i}\,p}{m\,\omega}\right)$$

$|n\rangle$

$$N\,|n\rangle=a^{\dagger}a|n\rangle=n|n\rangle$$

$$\begin{cases} a|n\rangle=\sqrt{n}\;|n-1\rangle \\ a^{\dagger}|n\rangle=\sqrt{n+1}\;|n+1\rangle \end{cases}$$

$a|0\rangle=0.$

$$x|n\rangle=\sqrt{\frac{\hbar}{2\,m\,\omega}}\big(\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle\big)$$

$$p|n\rangle=\mathrm{i}\,\sqrt{\frac{\hbar\,m\,\omega}{2}}\,\big(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle\big)$$

$$\langle m|x|n\rangle=\sqrt{\frac{\hbar}{2\,m\,\omega}}\,\left\{\sqrt{n}\,\delta_{n-1,m}+\sqrt{n+1}\,\delta_{m,n+1}\right\}$$

$$\langle m|x^2|n\rangle$$

$$\langle m|p^2|n\rangle$$