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The Harmonic Oscillator

$$H=\frac{p'^{\,2}}{2m}+\frac{1}{2}\,k\,x'^{\,2},\quad [x',p']=\mathrm{i}\hbar$$

Make the canonical transformation

$$\begin{cases} x' &=& x\sqrt{\frac{1}{m\,\omega}} \\ p' &=& p\sqrt{m\,\omega} \end{cases},\quad \omega=\sqrt{\frac{k}{m}}$$

We still have $[x,p]=\mathrm{i}\hbar$.

$$H=\frac{\omega}{2}\big(p^2+x^2\big)$$

$$a=(x+\mathrm{i}\,p)\,\frac{1}{\sqrt{2\hbar}},\quad a^{\dagger}=(x-\mathrm{i}\,p)\,\frac{1}{\sqrt{2\hbar}}$$

$$x=\big(a+a^{\dagger}\big)\sqrt{\frac{\hbar}{2}},\quad p=\frac{1}{\mathrm{i}}\,\big(a-a^{\dagger}\big)\,\sqrt{\frac{\hbar}{2}}$$

$$[a,a^{\dagger}]=\frac{1}{2\hbar}[x,-\mathrm{i}\,p]+\frac{1}{2\hbar}[\mathrm{i}\,p,x]=1$$

$$a^{\dagger}a=\frac{1}{2\hbar}(x-\mathrm{i}\,p)(x+\mathrm{i}\,p)=\frac{1}{2\hbar}\big(x^2+p^2-\mathrm{i}\,[p,x]\big)=\frac{1}{2\hbar}\big(x^2+p^2-\hbar^2\big)=\frac{1}{2\hbar}\big(x^2+p^2\big)-\frac{1}{2}$$

$$\hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right)=\frac{\omega}{2}\big(x^2+p^2\big)=H$$

$$H=\hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right)$$

The number operator $\mathbb{n}=a^{\dagger}a$.

$$[n,a]=\big[a^{\dagger}a,a\big]=a^{\dagger}[a,a]+\big[a^{\dagger},a\big]a=-a$$

$$\big[n,a^{\dagger}\big]=\big[a^{\dagger}a,a^{\dagger}\big]=a^{\dagger}$$

Assume $|n\rangle$ is an eigenvector of $\mathbb{n}$. $\mathbb{n}|n\rangle=n|n\rangle$

$$\mathbb{n}\big(a^{\dagger}|n\rangle\big)=a^{\dagger}a\,a^{\dagger}\,|n\rangle=a^{\dagger}\big(a^{\dagger}a+1\big)|n\rangle=a^{\dagger}(n+1)|n\rangle=(n+1)\,a^{\dagger}|n\rangle$$

$$\Rightarrow\quad a^{\dagger}|n\rangle\propto|n+1\rangle$$

$$\mathbb{n}(a|n\rangle)=a^{\dagger}a\,a|n\rangle=\big(a\,a^{\dagger}-1\big)\,a|n\rangle=\big(a\,a^{\dagger}\,a-a\big)\,|n\rangle=(n-1)\,a\,|n\rangle$$

$$\Rightarrow\quad a|n\rangle\propto|n-1\rangle$$

Since H is nonnegative, $|0\rangle$ must be an eigenket.

$$(\langle 0|a)\big(a^\dagger|0\rangle\big) = \langle 0|a^\dagger a + 1|0\rangle = \langle 0|\mathbb{n} + 1|0\rangle = 1$$

$$|1\rangle = a^\dagger|0\rangle$$

$$(\langle 1|a)\big(a^\dagger|1\rangle\big) = (1+1)\langle 1|1\rangle = 2$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\boxed{|n\rangle = \frac{\big(a^\dagger\big)^n}{\sqrt{n!}}\,|0\rangle}$$

$|0\rangle$

$\mathbb{n}|0\rangle = 0$ is a second order differential equation. $a|0\rangle = 0$ is a first order equation.

$$(x + \mathrm{i}p)\,|0\rangle = 0$$

$$\left(x + \mathrm{i}\,\frac{\hbar}{\mathrm{i}}\,\frac{\partial}{\partial x}\right)|0\rangle = 0$$

$$\left(x + \hbar\,\frac{\partial}{\partial x}\right)|0\rangle = 0$$

$$\langle x|\psi\rangle \propto \mathrm{e}^{-x^2/2\hbar}$$

$\langle 0|0\rangle = 1$ for the normalisation. [insert graph of $|0\rangle$, $|1\rangle$ and $|2\rangle$ here, $|n\rangle$ has n nodes].

$$\langle 2|x^3|3\rangle = \left(\frac{\hbar}{2}\right)^{3/2}\langle 2|\big(a + a^\dagger\big)^3|3\rangle =$$

$$= \left(\frac{\hbar}{2}\right)^{3/2}\langle 2|\underbrace{a^3}_{\rightsquigarrow 0} + a^2a^\dagger + a\,a^\dagger a + a^\dagger a\,a + a^\dagger a^\dagger a + a^\dagger a\,a^\dagger + a\,a^\dagger a^\dagger + \underbrace{\big(a^\dagger\big)^3}_{\rightsquigarrow 0}|3\rangle =$$

$$= \left(\frac{\hbar}{2}\right)^{3/2}\langle 2|a\,\big(a\,a^\dagger\big) + a\,\big(a^\dagger a\big) + \big(a^\dagger a\big)a|3\rangle =$$

$$= \left(\frac{\hbar}{2}\right)^{3/2}\langle 2|a(\mathbb{n}+1) + a\,\mathbb{n} + \mathbb{n}\,a|3\rangle = \left(\frac{\hbar}{2}\right)^{3/2}\langle 2|a\cdot 4 + a\cdot 3 + 2\,a|3\rangle =$$

$$= \left(\frac{\hbar}{2}\right)^{3/2}\cdot 9\,\langle 2|a|3\rangle = \left(\frac{\hbar}{2}\right)^{3/2} 9\,\sqrt{3}$$

Lagrangian

$$L(q,\dot{q}) = \text{kinetic energy} - \text{potential energy} = \frac{1}{2}m\dot{q}^2 - V(q)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}) + \frac{\partial V}{\partial q} = 0$$

$$m\,a = -\frac{\partial V}{\partial q} = F$$

Hamiltonian replace $q, \dot{q}$ with q, p.

$$p := \frac{\partial L}{\partial \dot{q}}$$

$$H(p,q) = p\,\dot{q} - L(q,\dot{q})$$

$$\mathrm{d}H = \mathrm{d}p\cdot\dot{q} + p\,\mathrm{d}\dot{q} - \mathrm{d}q\,\frac{\partial L}{\partial q} - \mathrm{d}\dot{q}\,\frac{\partial L}{\partial \dot{q}} = \mathrm{d}p\cdot\dot{q} + p\,\mathrm{d}\dot{q} - \mathrm{d}q\cdot\dot{p} - \mathrm{d}\dot{q}\cdot p$$

Now we arrive at Hamilton's equations:

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}.$$

Canonical transformation $(p,q), H \quad\rightarrow\quad P, Q, K$. Hamilton's equations with p, q and H are equivalent to Hamilton's equations with P, Q and K.

Let $F(q,P,t) =$ generating function. Definition:

$$\frac{\partial F}{\partial q} = p, \quad \frac{\partial F}{\partial P} = Q$$

$$L = p\,\dot{q} - H = P\dot{Q} - K \quad \text{— Not exactly:}$$

We can add a total time derivative to the Lagrangian without changing the equations of motion:

$$L = P\dot{Q} - K + \frac{\mathrm{d}}{\mathrm{d}t}(-PQ + F)$$

$$L = P\dot{Q} - K + \left(-\dot{P}Q - P\dot{Q} + \frac{\partial F}{\partial q}\,\dot{q} + \frac{\partial F}{\partial P}\,\dot{P} + \frac{\partial F}{\partial t}\right)$$

$$= P\dot{Q} - K + \left(-\dot{P}Q - P\dot{Q} + p\,\dot{q} + Q\,\dot{P} + \frac{\partial F}{\partial t}\right) = L = p\,\dot{q} - H$$

$$\Rightarrow K = H + \frac{\partial F}{\partial t}$$

This is our new Hamiltonian.

$K(P,Q) = 0$. Hamilton-Jacobi, use S instead of F.

$$p = \frac{\partial S}{\partial q}$$

Normally $S = S(q,\, t)$ with P implicit — P is just a constant anyway. The Hamiltion-Jacobi equation is

$$0 = H(p, q) + \frac{\partial S}{\partial t}$$

$$0 = \frac{p^2}{2m} + V(q) + \frac{\partial S}{\partial t}$$

$$\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = -\frac{\partial S}{\partial t}}$$

Hamilton-Jacobi equation is a differential equation for an unknown function S. There are similarities with the Schrödinger equation, but not quite.

$$\langle q|\psi\rangle = \sqrt{\rho}\exp\!\left(\frac{\mathrm{i}}{\hbar}S(q, t)\right)$$

Expand in orders of $\hbar$, and we find the Hamilton-Jacobi equation in the first order as $\hbar \to 0$.