2008–09–11

1.19

$$\langle (\Delta A)^2 \rangle \cdot \langle (\Delta B)^2 \rangle \geqslant \frac{1}{4} |\langle [A, B] \rangle|^2$$

$A \to S_x,\ B \to S_y.$

$$S_x = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \quad \Rightarrow \quad S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \Rightarrow \quad S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle + | S_x | + \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\langle + | S_x^2 | + \rangle = \frac{\hbar^2}{4}$$

$$\langle + | \, (\Delta S_x)^2 \, | + \rangle = \frac{\hbar^2}{4} - 0^2 = \frac{\hbar^2}{4}$$

$$\langle \, (\Delta S_y) \, \rangle = \frac{\hbar^2}{4}$$

$$\langle \, (\Delta S_x)^2 \, \rangle \langle \, (\Delta S_y)^2 \, \rangle = \frac{\hbar^4}{16}$$

$$[S_x, S_y] = \mathrm{i}\hbar \, S_z$$

$$[S_i, S_j] = \mathrm{i}\hbar \, \varepsilon_{ijk} \, S_k$$

$$\langle + | \, [S_x, S_y] \, | + \rangle = \mathrm{i}\hbar \, \frac{\hbar}{2}$$

$$\Rightarrow \frac{1}{4} | \, \langle \, [S_x, S_y] \rangle \, |^2 = \frac{1}{4} \frac{\hbar^4}{4} = \frac{\hbar^4}{16}$$

b) $A \to S_x, B \to S_y,\ |\alpha\rangle = |S_x; + \rangle = \frac{1}{\sqrt{2}} \left(| + \rangle + | - \rangle \right)$

$$|S_x; + \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\langle S_x; + | S_x | S_x; + \rangle = \frac{\hbar}{2}$$

$$\langle S_x; + | S_x^2 | S_x; + \rangle = \frac{\hbar^2}{4}$$

$$\langle \, (\Delta S_x)^2 \rangle = 0$$

$$\langle \, (\Delta S_y)^2 \, \rangle = \frac{\hbar^2}{4}$$

$$\frac{1}{4} |\langle S_x; + | \mathrm{i} \, \hbar \, S_z \, |S_x; + \rangle|^2 = \frac{\hbar^2}{4} \left| \frac{1}{2} \cdot \frac{\hbar}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2 = 0$$

1.23 $|1\rangle$, $|2\rangle$ and $|3\rangle$

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\mathrm{i}\, b \\ 0 & \mathrm{i}b & 0 \end{pmatrix}$$

a) $\det(B-\lambda I)=0$:

$$(b-\lambda)(\lambda^2-b^2)=0$$

$\lambda=b=\lambda_2$; $\lambda_3=-b$.

b) $[A,B]=0$. Is shown by multiplying them together.

c) $A\,u_i=\lambda_i\,u_i$, $B\,u_i=\tilde{\lambda_i}u_i$

$$u_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\x\\x\end{pmatrix}$$

$$u_2=\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{\mathrm{i}}{\sqrt{2}}\end{pmatrix},\quad u_3=\begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\-\frac{\mathrm{i}}{\sqrt{2}}\end{pmatrix}$$

$C=A+B$.

1.26

$$|a^{(i)}\rangle\rightarrow|b^{(i)}\rangle=U|a^{(i)}\rangle$$

$$U^\dagger U=UU^\dagger=I$$

$$U=\sum_r|b^{(r)}\rangle\langle a^{(r)}|$$

$$U\,|a^{(i)}\rangle=\sum_r|b^{(r)}\rangle\langle a^{(r)}|a^{(i)}\rangle=|b^{(i)}\rangle$$

$$\{|+\rangle,\quad|-\rangle\}$$

$$\{\quad|S_x;+\rangle,\quad|S_x;-\rangle\quad\}$$

$$U\,|\pm\rangle=|S_x;\pm\rangle$$

$$\langle+|U|+\rangle=\langle+|S_x;+\rangle=\langle+|\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)=\frac{1}{\sqrt{2}}$$

$$\langle+|U|-\rangle=\frac{1}{\sqrt{2}}$$

$$\langle-|U|+\rangle=\frac{1}{\sqrt{2}}$$

$$\langle-|U|-\rangle=-\frac{1}{\sqrt{2}}$$

$$U\doteq\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$U=\sum_r|b^{(r)}\rangle\langle a^{(r)}|=|S_x;+\rangle\langle+|+|S_x;-\rangle\langle-|\doteq\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$|\alpha\rangle$, $U^\dagger(\,\vdots\,)$, $U^\dagger(\,\vdots\,\vdots\,)U$

1.30

$$x|x'\rangle = x'|x'\rangle$$

$$T(\boldsymbol{l})|\boldsymbol{x'}\rangle = |\boldsymbol{x'}+\boldsymbol{l}\rangle$$

$$T(\boldsymbol{l}) = \exp\left(\frac{-\,\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right)$$

$\boldsymbol{p}$: momentum operator

$$[x_i, p_j] = \mathrm{i}\,\hbar\,\delta_{ij}$$

$$\begin{cases} [x_i, G(\boldsymbol{p})] = \mathrm{i}\hbar\,\frac{\partial G}{\partial p_i} \\ [p_i, F(\boldsymbol{x})] = -\,\mathrm{i}\hbar\,\frac{\partial F}{\partial x_i} \end{cases}$$

a) $[x_i, T(\boldsymbol{l})] =$

$$= \mathrm{i}\hbar\,\frac{\partial T}{\partial p_i} = \mathrm{i}\hbar\,\frac{\partial}{\partial p_i}\exp\left(\frac{-\,\mathrm{i}\,p_j l_j}{\hbar}\right) = l_i \exp\left(-\,\frac{\mathrm{i}\,p_j\,l_j}{\hbar}\right) = l_i\,T(\boldsymbol{l})$$

$$\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle$$

$$\rightarrow \langle\alpha|T^\dagger x_i T|\alpha\rangle = \langle\alpha|T^\dagger(T x_i + [x_i, T])|\alpha\rangle = \langle\alpha|x_i|\alpha\rangle + l_i\langle\alpha|\alpha\rangle = \langle x_i\rangle + l_i$$

$$\langle\boldsymbol{x}\rangle \rightarrow \langle\boldsymbol{x}\rangle + \boldsymbol{l}$$

1.29

$$[x_i, G(\boldsymbol{p})] = \mathrm{i}\,\hbar\,\frac{\partial G}{\partial p_i}$$

$$[p_i, F(\boldsymbol{x})] = -\,\mathrm{i}\,\hbar\,\frac{\partial F}{\partial x_i}$$

$$G(\boldsymbol{p}) = \sum_{n,m,l} a_{nml}\,p_x^n\,p_y^m p_z^l$$

$$[x, G(\boldsymbol{p})] = \sum_{n,m,l} a_{nml}[x, p_x^n]\,p_y^m\,p_z^l$$

$$[x_i, p_j^n] \propto \mathrm{i}\,\hbar\,n\,p_j^{n-1}\,\delta_{ij}$$

$$[x_i, p_j^n] = \left[x_i, p_j^{n-1}\right]p_j + p_j^{n-1}[x_i, p_j] = \left[x_i, p_j^{n-1}\right]p_j + \mathrm{i}\hbar\,p_j^{n-1}\delta_{ij} =$$

$$= \left[x_i, p_j^{n-2}\right]p_j^2 + p_j^{n-2}[x_i, p_j]p_j + \mathrm{i}\hbar\,p_j^{n-1}\delta_{ij} = \left[x_i, p_j^{n-2}\right]p_j^2 + 2\mathrm{i}\hbar\,p_j^{n-1}\delta_{ij}$$

$$[x_i, p_j^n] = \ldots = [x_i, p_j^0]p_j^n + n\,\mathrm{i}\,\hbar\,\delta_{ij} = n\,\mathrm{i}\,\hbar\,p_j^{n-1}\,\delta_{ij} = \mathrm{i}\hbar\,\frac{\partial p_j^n}{\partial p_i}$$

$$\left[x_i, p_i^n p_j^m p_k^l\right] = n\,\mathrm{i}\,\hbar\,p_i^{n-1} p_j^m p_k^l \text{ assuming } i \neq j, k$$

$$[x_i, G(\boldsymbol{p})] = \sum_{n,m,l} a_{nml}\,n\,\mathrm{i}\,\hbar\,p_i^{n-1}\,p_j^m\,{p_k}^l = \mathrm{i}\hbar\,\frac{\partial G}{\partial p_i}$$

b) $[x^2,p^2]=$

$$=\big[x,p^2\big]x+x\big[x,p^2\big]=p[x,p]x+[x,p]p\,x+x[x,p]p+x\,p[x,p]=$$

$$=2\,\mathrm{i}\,\hbar\,p\,x+2\,\mathrm{i}\,\hbar\,x\,p=2\,\mathrm{i}\,\hbar\,\{x,p\}$$

$[A,B]_{\mathrm{QM}}=\mathrm{i}\hbar\,[A,B]_{\mathrm{Poisson}}$.

$$\big[x^2,p^2\big]_{\mathrm{Poisson}}=\frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p}-\frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x}=2\,x\cdot 2\,p=4\,x\,p$$

1.31 $\langle\boldsymbol{x}\rangle\to\langle\boldsymbol{x}\rangle+\mathrm{d}\boldsymbol{x}'$, $\quad\langle\boldsymbol{p}\rangle\to\langle\boldsymbol{p}\rangle$.

$$|\alpha\rangle\to T(\mathrm{d}\boldsymbol{x}')\,|\alpha\rangle$$

$$[\boldsymbol{x},T(\mathrm{d}\boldsymbol{x}')]=\mathrm{d}\boldsymbol{x}'$$

$$[\boldsymbol{p},T(\mathrm{d}\boldsymbol{x}')]=0$$

$$\langle\boldsymbol{x}\rangle\quad\to\quad\langle\alpha|T^\dagger(\mathrm{d}\boldsymbol{x}')\boldsymbol{x}T(\mathrm{d}\boldsymbol{x}')|\alpha\rangle=$$

$$=\langle\alpha|T^\dagger(\mathrm{d}\boldsymbol{x}')\Big(T(\mathrm{d}\boldsymbol{x}')\boldsymbol{x}+\mathrm{d}\boldsymbol{x}'\Big)|\alpha\rangle=$$

$$=\langle\alpha|\boldsymbol{x}|\alpha\rangle+\langle\alpha|T(\mathrm{d}\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'|\alpha\rangle=\langle\boldsymbol{x}\rangle+\mathrm{d}\boldsymbol{x}'$$

$$\langle\boldsymbol{p}\rangle\to\langle\alpha|T^\dagger(\mathrm{d}\boldsymbol{x}')\boldsymbol{p}T(\mathrm{d}\boldsymbol{x}')|\alpha\rangle=\langle\alpha|\boldsymbol{p}|\alpha\rangle$$

$$\langle\boldsymbol{p}\rangle\quad\longrightarrow\quad\langle\boldsymbol{p}\rangle$$