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$$(\langle\psi|\gamma\rangle)\langle\tau| = \langle\psi|\,(|\gamma\rangle\langle\tau|)$$

Hilbert space of functions. $\langle$image of piecewise linear function changing direction at $x_0, \ldots, x_{N-1}\rangle$

$$|f\rangle = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix}, \quad |x_n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \longleftarrow \text{ position space basis.} \langle x_n|x_{n'}\rangle = \delta_{n,n'}$$

$$\langle x_n, f\rangle = f(x_n)$$

Position operator is diagonal:

$$x\begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} x_0 & & & \\ & x_1 & & \\ & & \ddots & \\ & & & x_{N-1} \end{pmatrix} \begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} x_0 f_0 \\ \vdots \\ x_n f_n \\ \vdots \end{pmatrix}$$

Check: $\langle x_n|X|f\rangle = x_n\, f(x_n)$.

$$g(x) = \begin{pmatrix} g(x_0) & & \\ & \ddots & \\ & & g(x_{N-1}) \end{pmatrix}$$

Recall “momentum operator”

$$p \text{ “=” } \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x}$$

$$p\,\psi(x) = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x}\psi$$

$$\langle x_n|p|f\rangle = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x} f\bigg|_{x_n} = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x} \langle x_n|f\rangle$$

$p =$ ”generator of translations”

$$f(x+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\partial^n}{\partial x^n} f(x) = \sum_{n=0}^{\infty} \left(\frac{a\,\mathrm{i}}{\hbar}\right)^n \frac{1}{n!}\, p^n\, f = \exp\!\left(\frac{\mathrm{i}\,a}{\hbar}\, p\right) f$$

$$T_a = \mathrm{e}^{\mathrm{i}pa/\hbar}$$

$$\langle x|T_a|f\rangle = f(x+a)$$

Momentum space eigenfunctions.

$$\mathbb{p}|p\rangle = p'|p\rangle$$

$$\langle x|\mathbb{p}|p'\rangle = p'\langle x|p'\rangle$$

$$\frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x}\langle x|p'\rangle = p'\langle x|p'\rangle$$

This is a differential equation for $\langle x|p'\rangle$.

$$\langle x|p'\rangle = \exp\!\left(\frac{\mathrm{i}}{\hbar}\, p'\, x\right) \cdot \text{normalization}$$

Get normalization through resolution of unity

$$\langle x|x'\rangle = \sum_p \langle x|p\rangle\langle p|x'\rangle = \sum_p \exp\!\left(\frac{\mathrm{i}}{\hbar}\,p\,x\right)\exp\!\left(-\frac{\mathrm{i}}{\hbar}\,p\,x'\right) \to N^2\int \mathrm{d}p\exp\!\left(\frac{\mathrm{i}}{\hbar}\,p\,(x-x')\right) =$$

$$= 2\pi\hbar\,N^2\int \mathrm{d}k\exp(2\pi\mathrm{i}\,k\,(x-x')) = N^2\cdot 2\pi\hbar\,\delta(x-x')$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\!\left(\frac{\mathrm{i}p}{\hbar}\,x\right)$$

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\!\left(-\frac{\mathrm{i}p}{\hbar}\,x\right)$$

Commutation relation between x, p.

$$\langle x_n|\mathbb{p}x|f\rangle = \langle x_n|\mathbb{p}|x\,f\rangle = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\,x\,f(x)\bigg|_{x_n}$$

$$(\langle x_n|\mathbb{x})\mathbb{p}|f\rangle = \langle x_n|\mathbb{x}\mathbb{p}|f\rangle = \frac{\hbar}{\mathrm{i}}\,x\,\frac{\partial}{\partial x}\,f(x)\bigg|_{x_n}$$

$$\langle x_n|\mathbb{p}\,\mathbb{x} - \mathbb{x}\mathbb{p}|f\rangle = \frac{\hbar}{\mathrm{i}}\,f(x) = \frac{\hbar}{\mathrm{i}}\langle x_n|f\rangle$$

$$[\mathbb{x},\mathbb{p}] = \mathbb{x}\mathbb{p} - \mathbb{p}\mathbb{x} = \mathrm{i}\hbar$$

$$\mathbb{x} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 100 \end{pmatrix}$$

$$\mathbb{p} = \frac{\hbar}{2\mathrm{i}}\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & \\ & & & \ddots \end{pmatrix}$$

Canonical commutation relations

$$[x_i, p_j] = \delta_{ij}\cdot\mathrm{i}\hbar$$

	Classical mechanics	Quantum mechanics
I	A state is defined by x and p.	State of the system $\|\psi(t)\rangle$
II	A classical dynamical variable $\Omega(x,p)$	Dynamical variables are operators
	$L_z = x\,p_y - y\,p_x$	$L_z = \frac{\hbar}{\mathrm{i}}\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right)$
III	Measurement $\langle\Omega\rangle = \Omega(x,p)$	Measuring Ω will yield an eigenvalue of Ω, say ω, with probability $\|\langle\omega\|\psi\rangle\|^2$. Afterward, the system will be in state $\|\omega\rangle$.

$$\langle\Omega\rangle = \sum_\omega \omega\,|\langle\omega|\psi\rangle|^2 = \sum_\omega \omega\,\langle\psi|\omega\rangle\langle\omega|\psi\rangle = \langle\psi|\underbrace{\left(\sum_\omega \omega\,|\omega\rangle\langle\omega|\right)}_{=\Omega}|\psi\rangle = \langle\psi|\Omega|\psi\rangle$$

	Clasical mechanics	Quantum mechanics
Dynamics	Hamilton's Equations	
	$\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$	$\mathrm{i}\hbar \frac{\partial}{\partial t} \lvert \psi \rangle = H \lvert \psi \rangle$

Measurement process with polarizers:

$$P_\theta = R(\theta) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^{-1}(\theta)$$

$$\lvert \omega \rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\Omega = \lvert \omega \rangle \langle \omega \rvert = \begin{pmatrix} \cos \\ \sin \end{pmatrix} \begin{pmatrix} \cos & \sin \end{pmatrix} = \begin{pmatrix} \cos^2 & \sin\cos \\ \sin\cos & \sin^2 \end{pmatrix}$$

Simultaneous measurement: Λ lets $\lvert \lambda \rangle$ through, Ω lets $\lvert \omega \rangle$ through.

$$\lvert \psi_{\text{out}} \rangle = \lvert \omega \rangle \langle \omega \lvert \lambda \rangle \langle \lambda \lvert \psi \rangle$$

Under what conditions will Ω not affect the result? $\Omega\Lambda \lvert \lambda \rangle = \Lambda\Omega \lvert \lambda \rangle$, $(\Omega\Lambda - \Lambda\Omega)\lvert \lambda \rangle = 0$.

$$[\Omega, \Lambda] = 0: \quad \text{simultaneous measurements always OK}$$

$$[\Omega, \Lambda] \lvert \psi \rangle = 0 \Rightarrow \lvert \psi \rangle = 0: \quad \text{measurement of } \Omega \text{ always affects the measurement of } \Lambda$$

$$[\Omega, \Lambda] \lvert \lambda_n \rangle = 0$$