2008–09–04

Recall polarizors ⟨fig1⟩

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \mathrm{e}^{\mathrm{i}(kz-\omega t)} \equiv |\psi\rangle\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}$$

$$|\psi\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi\rangle_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\psi\rangle_{\mathrm{c}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

Intensity? $= a^2 + b^2$? We have to take $|a^2| + |b^2|$.

"Bra" vector $\langle\psi| \sim (a^*, b^*)$.

$$\text{Intensity} = \langle\psi|\psi\rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2$$

Vector space $|\psi_x\rangle + |\psi_y\rangle = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = |\psi\rangle_{xy}$. Here we get $\langle\psi_{xy}|\psi_{xy}\rangle = 2$.

$I(|\psi_x\rangle + |\psi_x\rangle) = 4$.

$$|\psi'\rangle = R_\theta\, |\psi\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

⟨fig2⟩

$\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$ x-polarizers $|\psi\rangle_{\mathrm{out}}$

$\left(\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right)$ y-polarizers $|\psi\rangle_{\mathrm{out}}$ $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$Py \leftarrow Px$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Birefringent 1/4 wave plate $\left(\begin{smallmatrix} 1 & \\ & \mathrm{i} \end{smallmatrix}\right)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P_\theta = R(\theta) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} R^{-1}(\theta)$$

Operators transform under similarity transformations.

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}$$

$$P_\theta = \frac{1}{2} \begin{pmatrix} 1 + \cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1 + \cos 2\theta \end{pmatrix}$$

$$P_{45^\circ} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Hilbert-space, kets, bras, inner products, Hermitian adjoint, summation convention, dulity $\langle\alpha|A^\dagger \longleftrightarrow A|\alpha\rangle$. I don't write down everything he writes on the whiteboard now, because this is explained well in the book.

Unitary transformations: $U^\dagger U = 1$, where 1 is the identity matrix.

A unitary transformation preserves the inner product:

$$\langle \alpha' | \beta' \rangle = \sum_j \langle \alpha | U^\dagger)_j \, (U|\beta\rangle)_j = \langle \alpha | \big(U^\dagger U\big) | \beta \rangle$$

Orthonormal basis $|\alpha_j\rangle$:

$$\langle \alpha_j | \alpha_k \rangle = \delta_{jk}$$

Outer product:

$$|\beta\rangle\langle\alpha| = \begin{pmatrix} \beta_1 \alpha_1^* & \cdots & \beta_1 \alpha_n^* \\ \vdots & & \vdots \\ \beta_n \alpha_1^* & \cdots & \beta_n \alpha_n^* \end{pmatrix}$$

(Tensor product, "Kronecker product")

$$|\gamma'\rangle = (|\alpha\rangle\langle\beta|)\,|\gamma\rangle$$

$$|\gamma_i'\rangle = \sum_j \alpha_i \beta_j^* \gamma_j = \alpha_i \sum_j \beta_j^* \theta_j = a_i \langle \beta | \gamma \rangle$$

Associative: $|\alpha\rangle\langle\beta|\gamma\rangle$.

Relation between unitary transformation and orthonormal basis: assume $|\alpha_n\rangle$ is an orthonormal basis.

$$U_{nm} = (\alpha_n)_m$$

$$\big(U U^\dagger\big)_{ij} = U_{ik} U^\dagger_{kj} = \sum_k (\alpha_i)_k (\alpha_j^*)_k = \langle \alpha_j | \alpha_i \rangle = \delta_{ij}$$

$$U U^\dagger = \mathbb{1} \Longleftrightarrow U^\dagger U = \mathbb{1}$$

$$\mathbb{1} = \big(U^\dagger U\big)$$

$$(U^\dagger U)_{ij} = \sum_n \Big(U^\dagger_{in}\Big) U_{nj} = \sum_n (\alpha_n^*)_i (\alpha_n)_j = \sum_n (|\alpha_n\rangle\langle\alpha_n|)_{ij}$$

$$1 = \sum_n |\alpha_n\rangle\langle\alpha_n| \quad \Rightarrow \quad \text{Resolution of unity}$$

Linear algebra:

if H is Hermitian $H = H^\dagger$, then $\exists U$: $U^\dagger H U = \text{diagonal}$. $H = U D U^\dagger$.

Assume H is Hermitian. $|\alpha_i\rangle$ are the eigenvectors

$$H = \sum_n \varepsilon_n |\alpha_n\rangle\langle\alpha_n|$$

$$H|\alpha_i\rangle = \sum_n \varepsilon_n |\alpha_n\rangle\langle\alpha_n|\alpha_i\rangle = \sum_n \varepsilon_n |\alpha_n\rangle\,\delta_{ni} = \varepsilon_i |\alpha_i\rangle$$

Invariants.

$$\mathrm{Tr}(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ji} A_{ij} = \mathrm{Tr}(BA)$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$$

$$\mathrm{Tr}\big(U^\dagger H U\big), \quad \mathrm{Tr}\big(U U^\dagger H\big), \quad \mathrm{Tr}(H) = \sum \text{eigenvalues}$$

Eigenvalues are preserved under *all* unitary transformations.

$$H|\alpha\rangle = \varepsilon|\alpha\rangle$$

$$\underbrace{U^\dagger H U}_{=H'} U^\dagger|\alpha\rangle = U^\dagger \varepsilon|\alpha\rangle$$

$$H'\big(U^\dagger|\alpha\rangle\big) = \varepsilon\big(U^\dagger|\alpha\rangle\big)$$

$\mathrm{e}^{\mathrm{i}H}, \ln H, \tanh H$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$f(H) = \sum_{n=0}^{\infty} a_n H^n = a_0 1 + a_1 H + \frac{1}{2} H H + \frac{1}{3!} H \cdot H \cdot H =$$

$$= a_0 U U^\dagger + a_1 U U^\dagger H U U^\dagger + \frac{a_2}{2} U U^\dagger H U U^\dagger U U^\dagger + \cdots$$

$$= U\Big(a_0 \cdot 1 + a_1 D + \frac{a_2}{2} D^2 + \frac{a_3}{3!} D^3 + \cdots\Big) U^\dagger =$$

$$= U \begin{pmatrix} f(\varepsilon_1) & & & \\ & f(\varepsilon_2) & & \\ & & \ddots & \\ & & & f(\varepsilon_n) \end{pmatrix} U^\dagger = U f(D) U^\dagger$$