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Einstein:

$$E = h\nu = \frac{h}{2\pi} \cdot 2\pi\nu = \hbar\,\omega$$

Bohr:

$$L = n\,\hbar$$

$L = p\,r$

de Broglie:

Standing waves, the circumference is an integer number of wavelengths:

$$L = n\,\hbar = p\,r = p\,\frac{n\,\lambda}{2\pi} \quad \Rightarrow \quad p = \frac{2\pi\hbar}{\lambda}$$

Schrödinger:

$$\psi = \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\text{: solution}$$

$$p\psi = \hbar k\,\psi = \frac{\hbar}{\mathrm{i}}\,\frac{\partial}{\partial x}\,\psi$$

$$\frac{p^2}{2m} = \ldots = -\,\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial x^2}\psi$$

$$E\psi = \hbar\,\omega\,\psi = \mathrm{i}\,\hbar\,\frac{\partial}{\partial t}\,\psi$$

$$E = \frac{p^2}{2m} + V$$

$$\mathrm{i}\hbar\,\frac{\partial}{\partial t}\,\psi = \left(-\,\frac{\hbar^2}{2m}\cdot\frac{\partial^2}{\partial x^2} + V\right)\psi$$

Canonical substitution

$$\begin{aligned} E &\mapsto \mathrm{i}\,\hbar\,\frac{\partial}{\partial t} \\ p_x &\mapsto \frac{\hbar}{\mathrm{i}}\,\frac{\partial}{\partial x} \end{aligned}$$

Take the classical equation and substitute the above to get the quantum version.

What do we do in electromagnetism?

$$\omega = c\,|k|$$

$$\omega^2 = c^2\,\boldsymbol{k}^2$$

$$\psi = \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$

$$-\,\frac{\partial^2}{\partial t^2}\psi = -\,c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\psi = 0$$

$$\boldsymbol{E} = \mathrm{Re}(\psi)$$

ψ can be taken as a real field, because it is governed by a real equation. The Schrödinger equation, on the other hand, has a necessarily complex solution. The real part cannot be interpreted as a density, because it becomes negative. The correct interpretation is

$$\rho = |\psi|^2.$$

We demand that ρ obeys the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

$$\frac{\partial}{\partial t}\,\rho = \frac{\partial}{\partial t}\,\psi^* \psi = \psi^* \frac{\partial}{\partial t} \psi + \text{complex conjugate} =$$

$$= \frac{\psi^*}{\mathrm{i}\hbar} \cdot \mathrm{i}\hbar \frac{\partial}{\partial t} \psi + \text{complex conjugate} = [\text{Schrödinger}] =$$

$$= \frac{\psi^*}{\mathrm{i}\hbar} \left(-\frac{\hbar^2}{2m}\,\nabla^2 \psi + V\psi \right) + \text{complex conjugate} =$$

$$= -\frac{\hbar}{2\,m\,\mathrm{i}}\,\nabla \cdot (\psi^* \nabla \psi) + \text{complex conjugate} =$$

$$\frac{\partial \rho}{\partial t} = -\frac{\hbar}{2\,m\,\mathrm{i}}\,\nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

$$\boldsymbol{J} = \frac{\hbar}{2m\,\mathrm{i}} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

The equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

$$\int \frac{\partial \rho}{\partial t}\,\mathrm{d}^3 x + \int \nabla \cdot \boldsymbol{J}\,\mathrm{d}^3 x = 0$$

$$\frac{\partial}{\partial t} \int \rho\,\mathrm{d}^3 x + \oint_{\partial V} -\boldsymbol{J} \cdot \mathrm{d}\boldsymbol{A} = 0$$

If $\boldsymbol{J} = \mathbf{0}$ at the surface the latter term vanishes.

$$\frac{\partial}{\partial t} Q_{\text{tot}} = 0$$

Polarization:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathrm{e}^{\mathrm{i}(kz - \omega t)} \begin{pmatrix} a \\ b \end{pmatrix} = \mathrm{e}^{\mathrm{i}(kz - \omega t)}\,|\psi\rangle$$

$$|\psi\rangle_{x\,\text{polarization}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle_{y\,\text{polarization}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle_{\text{circular}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$