2009–03–06 Lecturer: Per Salomonsson

$$\varphi = \begin{pmatrix} \varphi^0 \\ \varphi^- \end{pmatrix}, \quad \delta\varphi = \frac{\mathrm{i}}{2}(g\,\boldsymbol{\tau}\cdot\boldsymbol{\Lambda} - g'\,\Lambda)\varphi$$

$$\tilde{\varphi} = -\mathrm{i}\tau_2\varphi^*, \quad \delta\tilde{\varphi} = \frac{\mathrm{i}}{2}(g\,\boldsymbol{\tau}\cdot\boldsymbol{\Lambda} + g'\,\Lambda)\tilde{\varphi}$$

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi + \mathcal{L}_\varphi = \mathcal{L}_A + (\mathcal{L}_{\psi k} + \mathcal{L}_{\psi y}) + (\mathcal{L}_{\varphi k} + \mathcal{L}_{\varphi v})$$

First generation: u, d, e, ν_e. Pure gauge piece, fermions with covariant derivatives, Yukawa, scalar field with covariant derivative, potential for φ.

$$-\mathcal{L}_{\phi v} = -\mu^2\phi^\dagger\phi + \frac{1}{2}\lambda\left(\phi^\dagger\phi\right)$$

Classically, the lowest energy state

$$\phi = \phi_0 = \text{const}; \quad -\left.\frac{\partial\mathcal{L}}{\partial\varphi^\dagger}\right|_{\phi=\phi_0} = \phi_0\Big(-\mu^2 + \lambda\,\phi_0^\dagger\phi_0\Big) = 0$$

In quantum theory: $\langle 0|\varphi|0\rangle = \phi_0$. $\phi_0^\dagger\phi_0 = \mu^2/\lambda$. $\phi = \phi_0 + \varphi$. As we will see the direction of ϕ_0 will define the photon field direction in SU(2) $\times$ U(1) space. By SU(2) symmetry ϕ^0 can be rotated to

$$\phi^0 = \frac{1}{\sqrt{2}}\begin{pmatrix} c \\ 0 \end{pmatrix}, \quad c > 0$$

This will make the carges correct for

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}.$$

The ϕ_0 part of $\mathcal{L}_{\phi,\text{kinetic}}$ will give masses to W and Z.

$$(\mathrm{D}^\mu\phi_0)^\dagger\,\mathrm{D}_\mu\phi_0 \equiv |\mathrm{D}_\mu\,\phi_0|^2 = \frac{1}{2}\left|\frac{1}{2}(g\,\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu - g'\,B')\begin{pmatrix} c \\ 0 \end{pmatrix}\right|^2 =$$

$$= \frac{1}{8}\,c^2\left[(g\,\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu - g'\,B_\mu)^2\right]_{11} = \frac{1}{8}\,c^2\Big[g^2\big(A_{1\mu}^2 + A_{2\mu}^2\big) + (g\,A_{3\mu} - g'\,B_\mu)^2\Big]$$

$$M_W^2 = \frac{1}{4}\,c^2\,g^2, \quad M_Z^2 = \frac{1}{4}\,c^2\left(g^2 + g'^2\right), \quad M_{(\gamma)}^2 = 0$$

$$Z_\mu = \frac{g\,A_{3\mu} - g'\,B_\mu}{\sqrt{g^2 + g'^2}}$$

$$A_\mu^{(\gamma)} = \frac{g'\,A_{3\mu} + g\,B_\mu}{\sqrt{g^2 + g'^2}}$$

$$W_\mu^\mp = \frac{1}{\sqrt{2}}(A_{1\mu} \pm \mathrm{i}\,A_{2\mu})$$

$\mathcal{L}_\phi$?

$$\phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$$

has four real components. Three are eaten by $W^{\pm}$, Z.

Parametrisation of $\phi(x) =$

$$= \mathrm{e}^{\frac{\mathrm{i}}{2}\boldsymbol{\tau}\cdot\boldsymbol{\theta}(x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}(c + \varphi(x))$$

$\boldsymbol{\theta}$: just a gauge transformation. $\varphi(x)$: the only physical scalar higgs particle.

$$\mathcal{L}_{\varphi} = \frac{1}{2}\,\partial^{\mu}\varphi\,\partial_{\mu}\varphi + \frac{1}{8}(c+\varphi)^2\left(g^2\big(W_{1\mu}^2 + W_{2,\mu}^2\big) + \big(g^2 + g'^2\big)Z_{\mu}^2\right) + \frac{1}{2}\mu^2(c+\varphi)^2 - \frac{1}{8}\lambda(c+\varphi)^4$$

$\mathcal{L}_{\psi Y}$ is the most gneral renormalizable SU(3) × SU(2) × U(1) invariant combination of therms of type "$\bar{\psi}\psi\phi$".

$$\mathcal{L}_{\psi W} = -\frac{\sqrt{2}}{c}\left[m_{\nu}\underbrace{\bar{q}_L}_{-\frac{1}{6}}\underbrace{\phi}_{-\frac{1}{2}}\underbrace{u_R}_{\frac{2}{3}} + m_d\underbrace{\bar{q}_L}_{-\frac{1}{6}}\underbrace{\tilde{\phi}}_{+\frac{1}{2}}\underbrace{d}_{-\frac{1}{3}} + m_{\nu}\underbrace{\bar{\psi}_L}_{\frac{1}{2}}\underbrace{\phi}_{-\frac{1}{2}}\nu_e + m_e\bar{\psi}_L\tilde{\phi}\;e\right] +$$

hermitian conjugate.

Neutrino mass term:

$$\psi_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$$

$\nu_{eL} \equiv -\mathrm{i}\,\tau_2\,\nu_R'^{\,*}$.

Both neutrino mass terms combine into neutrino Majorana mass matrix:

$$\begin{pmatrix} \nu_R^T & \nu_R'^{\,T} \end{pmatrix}\frac{\mathrm{i}}{2}\sigma^2\begin{pmatrix} M_{\nu} & m_{\nu} \\ m_{\nu} & 0 \end{pmatrix}\begin{pmatrix} \nu_R \\ \nu_R' \end{pmatrix} + \text{hermitian conjugate}$$

Diagonalise to get the physical masses. If $M_{\nu} \gg m_{\nu}$ then the eigenvalues are $M_{\nu}, -m_{\nu}^2/M_{\nu}$.

$\mathcal{L}_{\psi Y}$ Fermion covariant derivatives give weak and electromagnetic interactions of quarks and leptons.

$$\mathcal{L}_{\psi k} = \bar{q}_L\,\mathrm{i}\left(\partial\!\!\!/ + A\!\!\!/^{(c)} + \frac{\mathrm{i}}{2}\,g'\boldsymbol{\tau}\cdot\boldsymbol{A\!\!\!/} + \frac{\mathrm{i}}{6}\,g'\,B\!\!\!/\right)q_L$$

$$\bar{u}_R\,\mathrm{i}\left(\partial\!\!\!/ + A\!\!\!/^{(c)} + \frac{2}{3}\,\mathrm{i}\,g'\,B\!\!\!/\right)u_R$$

$$\bar{d}_R\left(\partial\!\!\!/ + A\!\!\!/^{(c)} - \frac{1}{3}\,\mathrm{i}\,g'\,B\!\!\!/\right)d_R$$

$$\bar{\psi}_L\mathrm{i}\left(\partial\!\!\!/ + \frac{\mathrm{i}}{2}\,g\,\boldsymbol{\tau}\cdot\boldsymbol{A\!\!\!/} - \frac{1}{2}\,g'\,B\!\!\!/\right)\psi_L$$

$$\bar{\nu}_R\,\mathrm{i}\,\partial\!\!\!/\nu_R$$

$$\bar{e}_R\,\mathrm{i}\big(\partial\!\!\!/ - \mathrm{i}\,g'\,B\!\!\!/\big)e_R$$

$$A_3 = \frac{g\,Z + g'\,A^{(\gamma)}}{\sqrt{g^2 + g'^2}}, \quad B = \frac{-g'\,Z + g\,A^{(\gamma)}}{\sqrt{g^2 + g'^2}}, \quad A_1 \pm \mathrm{i}\,A_2 = \sqrt{2}W^{\mp}$$

Example:

$$-\,\bar{q}_L\left(\frac{1}{2}\,g\,\boldsymbol{\tau}\cdot\mathbf{A}+\frac{1}{6}\,g'\,B\right)q_L-\bar{u}_R\,\frac{2}{3}\,g'\,\not{B}u_R+\not{d}_R\frac{1}{3}\,g'\,\not{B}\,d_R$$

$$=-\,\bar{u}_L\,\frac{g}{\sqrt{2}}\,\not{W}^+\,d_L-\bar{d}_L\,\frac{g}{\sqrt{2}}\not{W}-u_L$$

$$=-\,\frac{g\,g'}{\sqrt{g^2+g'^2}}\left(\frac{2}{3}\,\bar{u}\,\not{A}^{(\gamma)}u-\frac{1}{3}\,\bar{d}\,\not{A}^{(\gamma)}d\right)+Z\,\text{terms}$$

Theorem: an arbitrary matrix M can be diagonalized by two unitary transformations: $M = U_1^\dagger\,D\,U_2$. Proof: $M\,M^\dagger=U_1^\dagger D^2 U_1\Rightarrow\left(D^{-1}U_1M\right)\left(D^{-1}U_1M\right)^\dagger=()\Rightarrow D^{-1}U_1M=U_2$