2009–02–25

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More items from chapter 5.

Crossing symmetry.

Ex: $e\,\bar{e} \to \mu\bar{\mu}$

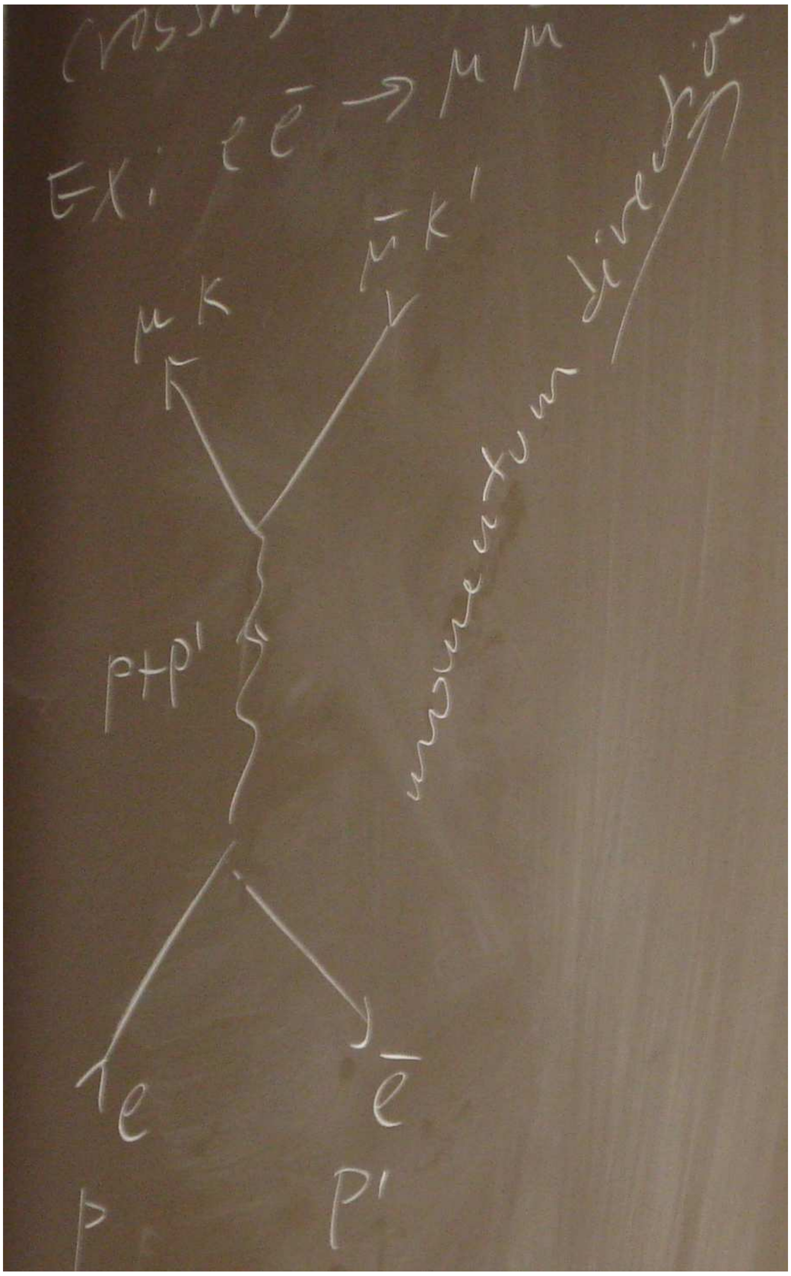

Figure 1.

$$\sum_{\text{spin}} |\mathcal{M}|^2 \sim \mathrm{Tr}\big((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big)\,\frac{(e^2)^2}{\big((p+p')^2\big)^2}\,\mathrm{Tr}\,\big((\not{k} + m)\gamma_\mu\,(k' - m)\gamma_\nu\big)$$

Compare $e+\mu \to e+\mu$

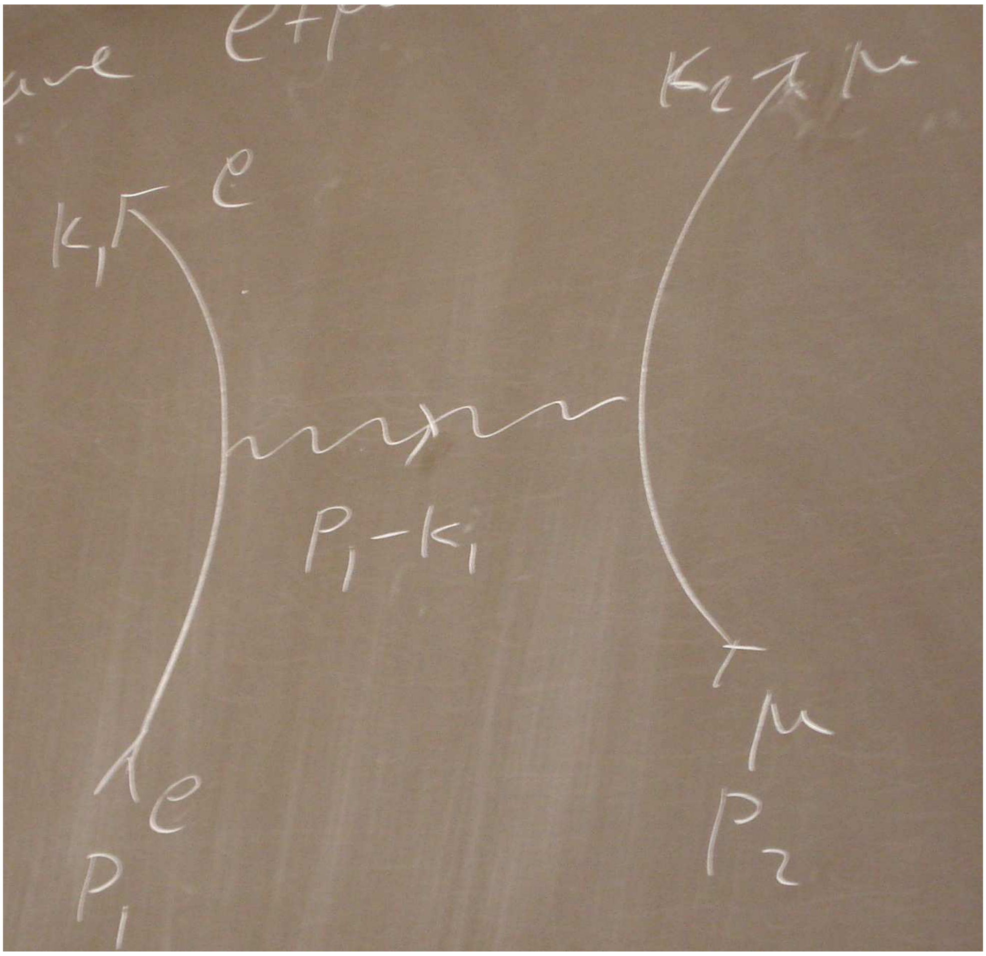

Figure 2. $e+\mu \to e+\mu$

The diagrams look very similar, although it is a different process. An incoming positron in the first case corresponds to an outgoing electron in the second process.

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \mathrm{Tr}\left(\left(\not{k}+m\right)\gamma_\mu\left(\not{p}+m\right)\gamma_\nu\right)\frac{e^4}{\left((p_1-k_1)^2\right)^2}\,\mathrm{Tr}\Big(\left(\not{k}_2+m\right)\gamma^\mu\Big(\not{p}_2+m\Big)\gamma^\nu\Big)$$

Momenta related by

$$\begin{array}{rcl} -p' & \leftrightarrow & k_1 \\ p & \leftrightarrow & p_1 \\ k & \leftrightarrow & k_2 \\ -k' & \leftrightarrow & p_2 \end{array}$$

By these transformations of momenta, one $\sum_{\text{spin}} |\mathcal{M}|^2$ is transformed to the other. In general, for $\sum_{\text{spin}} |\mathcal{M}|^2$ you have also to add a minus sign for each “crossed” fermion. In this way different processes are related: Compton scattering, $e+\gamma \to e+\gamma$ related to $e + \bar{e} \to \gamma + \gamma$

(matter annihilation into light) and $\gamma + \gamma \rightarrow e + \bar{e}$ (creation of matter from pure light).

Mandelstam variables s, t, u

Convenient set of scalar variables for 2 particle scattering. As an example, consider the simplest Quantum Field Theory, scalar $\frac{1}{6}\lambda\varphi^3$.

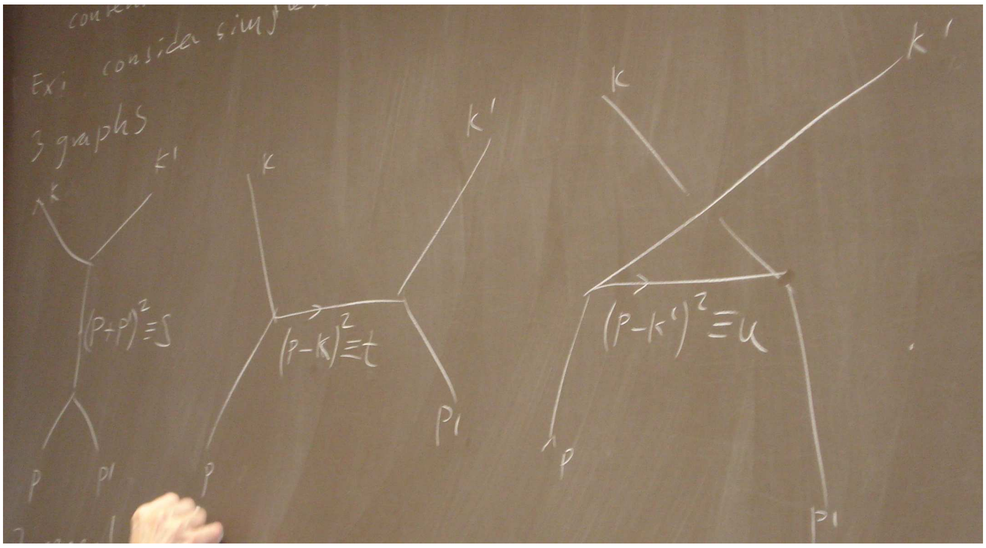

Figure 3.

In general $\mathcal{M} = f(p, p', k, k')$. Momentum conservation $p + p' = k + k'$. $p^2 = m^2$ etc. Three independent scalar variables: $p \cdot p', p \cdot k, p' \cdot k$.

In fact the Mandelstam varables is overcomplete set

$$s + t + u = 3p^2 + p'^2 + k^2 + k'^2 + 2p \cdot (p' - k - k') = p^2 + p'^2 + k^2 + k'^2 = \text{constant}$$

Ways of evaluating $|\mathcal{M}|^2$.

Trace technology method

$$\sum_s u(p)\,\bar{u}\,(p) = \not{p} + m, \quad \text{etc.}$$

This has the advantage that it has manifest relativistic symmetry. (The trace is relativistically invariant.) It is simple in simple situations.

Alternative way: explicit expressions for $u(p)$ etc. Advantage: easier in complicated situations.

Peskin&Schroeder illustrate it to explain the angular dependence of $|\mathcal{M}|^2$ for $e\ \bar{e} \ \rightarrow \ \mu\bar{\mu}$. Remember

$$\frac{1}{4}\sum |\mathcal{M}|^2 = e^2\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

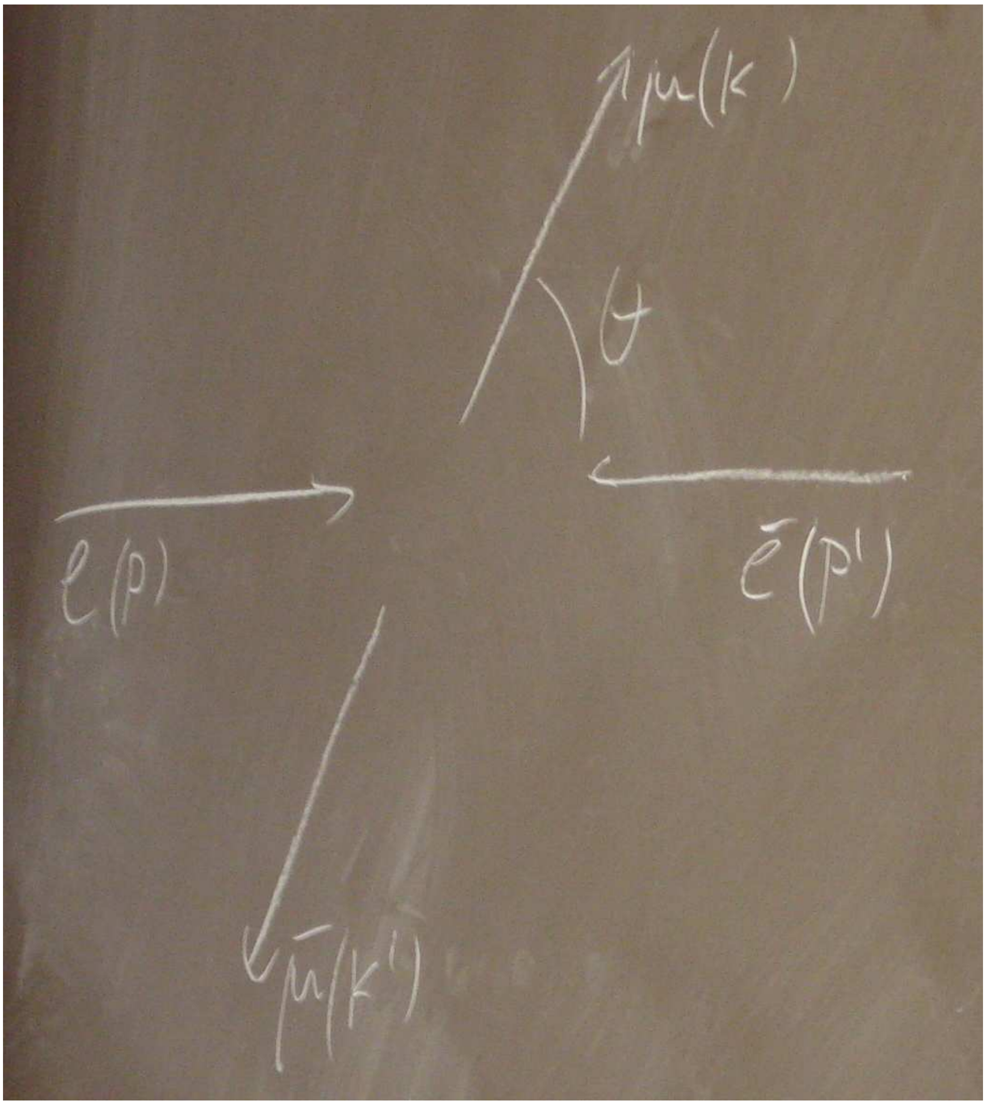

Figure 4.

$$p=(E,0,0,E), \quad p'=(E,0,0,-E)$$

$$u(p)=\begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi_e \\ \sqrt{p\cdot\bar{\sigma}}\;\xi_e \end{pmatrix}$$

$$p\cdot\sigma=E-\sigma_3 E=\begin{pmatrix} 0 & \\ & 2\,E \end{pmatrix}$$

$$u(p)=\begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi_e \\ \sqrt{p\cdot\bar{\sigma}}\;\xi_e \end{pmatrix}=\sqrt{2E}\begin{pmatrix} 0 \\ \xi_2 \\ \xi_1 \\ 0 \end{pmatrix}$$

$$v(p')=\begin{pmatrix} \sqrt{p'\cdot\sigma}\;\eta_{\bar{e}} \\ -\sqrt{p'\cdot\bar{\sigma}}\;\eta \end{pmatrix}=\sqrt{2E}\begin{pmatrix} \eta_1 \\ 0 \\ 0 \\ -\eta_2 \end{pmatrix}$$

$$\gamma^0\,\gamma^\mu=\begin{pmatrix} \bar{\sigma}^{\,\mu} & \\ & \sigma^\mu \end{pmatrix}$$

$$(\bar{v}\,(p')\,\gamma^\mu\,u(p))_e=v^\dagger(p')\gamma^0\gamma^\mu u(p)=2\,E\left[\overset{4}{(\,\eta_1^*\;\;0\,)}\bar{\sigma}^{\,\mu}\begin{pmatrix} 0 \\ \xi_2 \end{pmatrix}+(0,-\eta_2^*)\,\sigma^\mu\begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}\right]=$$

$$=2\,E\,[\eta_1^*\xi_2(0,-1,\mathrm{i},0)+\eta_2^*\xi_1(0,1,\mathrm{i},0)]$$

$\mu\bar{\mu}$ part of $\mathcal{M}$: First let μ also be extreme relativistic. Obtain it by rotating electron part by an angle θ. e.g. $(0,-1,\mathrm{i},0)\rightarrow(0,-\cos\theta,\mathrm{i},-\sin\theta)$, $(0,1,\mathrm{i},0)\rightarrow(0,\cos\theta,\mathrm{i},\cos\theta)$.

Then: $(0,-1,\mathrm{i},0)\cdot('')^* = -(1+\cos\theta)$. There are in total 4 nozero terms corresponding to:

$$\begin{array}{rccccccl} & e & \bar{e} & \rightarrow & \mu & \bar{\mu} & \text{Amp} \\ & - & + & & - & + & (1+\cos\theta) \\ \text{helicity:} & + & - & & + & - & (1+\cos\theta) \\ & - & + & & + & - & (1-\cos\theta) \\ & + & - & & - & + & (1-\cos\theta) \end{array}$$

If instead $\mu\,\bar{\mu}$ are nonrelativistic (i.e. $E = m_\mu + \varepsilon$)

$$u(k) = \sqrt{m_\mu}\begin{pmatrix} \xi \\ \xi \end{pmatrix}_\mu, \quad v(k') = \sqrt{m_\mu}\begin{pmatrix} \eta \\ -\eta \end{pmatrix}_{\bar{\mu}}$$

$$\mathcal{M} = (-\mathrm{i}\,e)^2 \frac{-\mathrm{i}}{(2\,E)^2}\,\bar{v}\,(p')\,\gamma^\mu\,u(p)\,\bar{u}\,(k')\,\gamma_\mu v(k')$$

$$\bar{v}\,(p')\gamma^\mu u(p) = 2\,E[\ldots(0,1,-\mathrm{i},0) + \ldots(0,1,\mathrm{i},0)]$$

$$\bar{u}\,(k)\,\gamma^\mu\,v(k') = m\,(\xi^\dagger,\xi^\dagger)\begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix}\begin{pmatrix} \eta \\ -\eta \end{pmatrix} = \begin{cases} 0, & \mu = 0 \\ -2\,m_\mu\,\xi^\dagger\sigma^i\eta, & \text{if } \mu = i \end{cases}$$

$$\mathrm{i}\,\mathcal{M} = \mathrm{i}\,e^2\Big[\ldots\,2\xi_\mu^\dagger\,\sigma^-\eta_\mu + \ldots\xi_\mu^\dagger\sigma^\dagger\eta_\mu\Big]$$

Features:

- pure s-wave
- $\neq 0$ only if $s_z = \frac{1}{2}$ for $e, \bar{e}\,, \mu, \bar{\mu}$ or $s_z = -\frac{1}{2}$ for $e, \bar{e}\,, \mu, \bar{\mu}\,.$

Last item: $e\,\bar{e} \rightarrow \mu\bar{\mu}$ boud state. $E = m_\mu - \varepsilon.$

$\mu\bar{\mu}$ use relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and centre of mass coordinate $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2).$

$$\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) \Rightarrow \psi(\boldsymbol{R},\boldsymbol{r}) \approx \psi(\boldsymbol{r}) = (2\pi)^{-3}\int \mathrm{d}^3k\,\tilde{\psi}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$|B\rangle = \frac{\sqrt{2M}}{\sqrt{2m_\mu}\,\sqrt{2m}}\int \frac{\mathrm{d}^3k}{(2\pi)^3}|k\uparrow,-k\uparrow\rangle\tilde{\psi}(k)$$

$$\mathcal{M}(\uparrow.\uparrow \rightarrow B) = \sqrt{\frac{2\,M}{2m\,2m}}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{\psi}(k)^*\,2e^2 = \sqrt{\frac{2}{M}}\,2\,e^2\,\psi(0)^*$$

$$\sigma(e^+e^- \rightarrow B) = \frac{1}{2}\,\frac{1}{2m}\,\frac{1}{2m}\underbrace{\int \frac{\mathrm{d}^3k_B}{(2\pi)^3}\,\frac{1}{2E_B}\,(2\pi)^4\,\delta(p + p' - K)}_{2\pi\delta(E_{\mathrm{cm}}^2 - M_B^2)} \times \frac{2}{\mathcal{M}}\,(2e^2)^2\,\frac{1}{2}|\psi(0)|^2 =$$

$$= \frac{2e^4}{\mathcal{M}^3}|\psi(0)|^2\,(2\pi)\,\delta(E_{\mathrm{cm}}^2 - M^2)$$

Infinite for $E_{\mathrm{cm}} = M$. Incompatible with unitarity.

$$\sigma(e^+e^- \rightarrow B) \neq 0 \quad \Rightarrow \quad \Gamma(B \rightarrow e^+e^-) \neq 0$$

and these processes are related by the crossing symmetry (same $|\mathcal{M}|^2$). Peskin&Schroeder:

$$\Gamma(B \to e^+ e^-) = \frac{e^4}{3\pi} \frac{|\psi(0)|^2}{M^2}$$

$\Rightarrow |B\rangle$ is not an energy eigenstate.

$$|B(t)\rangle \sim \mathrm{e}^{-\mathrm{i}t(M - \mathrm{i}\Gamma/2)} \quad \Rightarrow \quad ||B(t)\rangle|^2 \sim e^{-\Gamma t}$$

$$\Rightarrow \pi\,\delta(E^2 - M^2) = -\,\mathrm{Im}\frac{1}{E^2 - M^2 + \mathrm{i}\varepsilon} \to -\,\mathrm{Im}\frac{1}{E^2 - M^2 + \mathrm{i}\,M\,\Gamma} = \frac{M\,\Gamma}{(E^2 - M^2)^2 + M^2\Gamma^2}$$

$$\sigma \to 4\pi\,\frac{3\Gamma}{M}\,\frac{M\,\Gamma}{E^2 - M^2 + M^2\Gamma^2}$$

from relation between σ and Γ according to Peskin&Schroeder.

$$\sigma_{\max} = 4\pi\,\frac{3}{M^2}$$

compatible with partial wave unitarity.