2009–02–20

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Last time: $e^- + e^+ \to \mu^- + \mu^+$. This is the simplest kind, since we have different fermions. We only get one diagram. Compare this with Bhabha scattering and Møller scattering. Bhabha scattering is the electron–positron scattering process: $e^- + e^+ \to e^- + e^+$. Møller scattering is electron–electron scattering. In both these cases, there are two leading-order Feynman diagrams contributing to the interaction (figure 1).

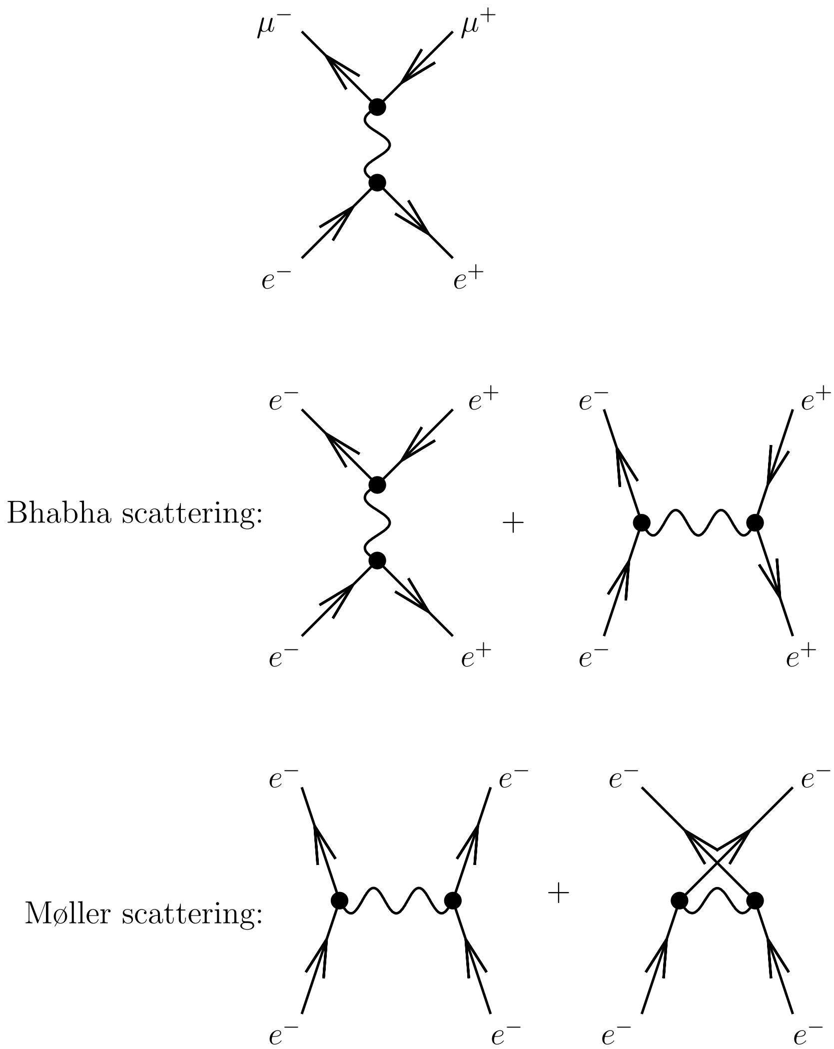

Figure 1. In Bhabha and Møller scattering, we need to take two diagrams into account. [In the second diagram of Bhabha scattering, the one without annihilation, Gabriele had both arrows going in the same direction, but surely this is the way it has to be.]

Compton scattering:

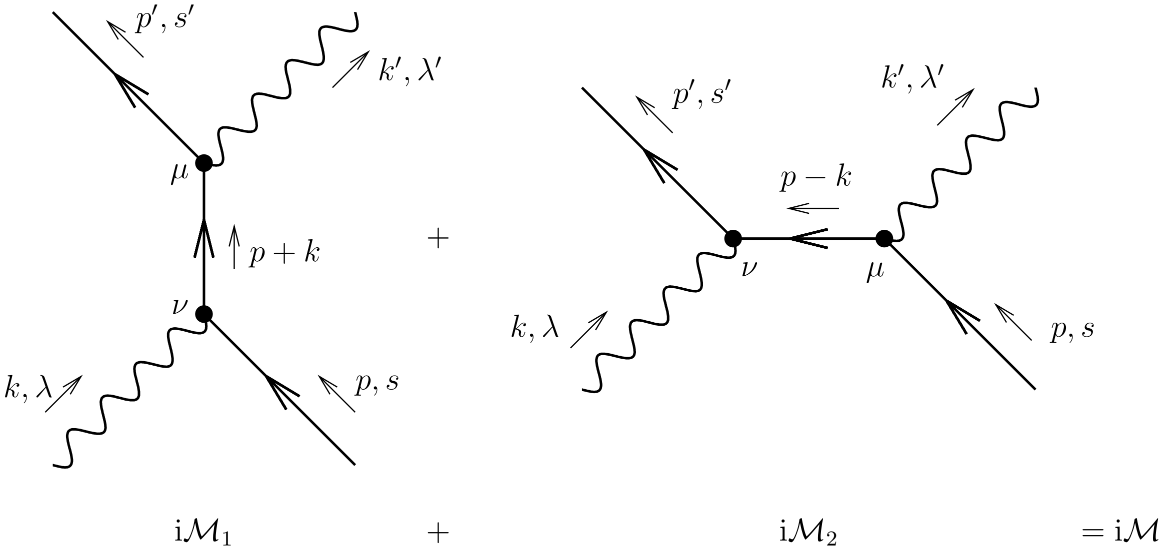

Figure 2. Compton scattering has two leading-order Feynman diagrams. λ and λ' denote the helicity of the respective photon.

$$\mathrm{i}\mathcal{M}_1 + \mathrm{i}\,\mathcal{M}_2 = \mathrm{i}\,\mathcal{M}$$

$$\mathrm{i}\,\mathcal{M}_1 = \bar{u}_{s'}(p')\,(-\mathrm{i}\,e\,\gamma^\mu)\,\frac{\mathrm{i}(\not{p} + \not{k} + m)}{(p+k)^2 - m^2}\,(-\mathrm{i}\,e\,\gamma^\nu)\,u_s(p)\,\varepsilon_{\nu\lambda}(k)\,\varepsilon^*_{\mu\lambda'}(k')$$

For a photon you have to take the polarisation into account, that's the ε above.

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \Rightarrow \quad \Box A_\nu - \partial_\nu\,\partial_\mu\,A^\mu = 0$$

Compare $\Box\phi = 0$, $\phi(x) = \mathrm{e}^{\pm\mathrm{i}k\cdot x}$, $k^2 = 0$. Ansatz: $A_\mu(x) = \varepsilon_\mu(k)\,\mathrm{e}^{\pm\mathrm{i}kx}$. You get:

$$-k^2\varepsilon_\mu + k_\mu\,k\cdot\varepsilon = 0.$$

(For the Dirac case, $\not{\partial}\psi$, $\psi = u\,\mathrm{e}^{\pm\mathrm{i}k\cdot x}$, $\not{p}u = 0$.)

1) ε_μ is parallel to k_μ: we get $\varepsilon_\mu = A\cdot k_\mu$. This is just a gauge transformation.

2) ε_μ is not parallel to k_μ: $k^2 = 0$, $k\cdot\varepsilon = 0$. This is the physical solution. It tells us that the photon is massless, and that the polarisation is orthogonal to the momentum.

Assume $\varepsilon\cdot k = 0$ for some $\varepsilon_\mu = (\varepsilon_0, \boldsymbol{\varepsilon}) \to (0, \boldsymbol{\varepsilon}')$. $k = (\varepsilon_0, \boldsymbol{k})$ with $|\boldsymbol{k}| = \varepsilon_0$. $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \boldsymbol{k}$. This is transverse gauge, and we get $\boldsymbol{k}\cdot\boldsymbol{\varepsilon}=0$. There are two [linearly independent] solutions $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$.

$$\varepsilon_\mu^{\lambda=1}=\begin{pmatrix}0\\ \boldsymbol{\varepsilon}_1\end{pmatrix},\quad \varepsilon_\mu^{\lambda=2}=\begin{pmatrix}0\\ \boldsymbol{\varepsilon}_2\end{pmatrix},\quad k^2=0,\quad \boldsymbol{k}\cdot\boldsymbol{\epsilon}_{1,2}=0$$

$$H\sim\int \mathrm{d}^4x\,\bar{\psi}\psi\cdot\phi\to\int \mathrm{d}^4x\,\bar{\psi}\,\gamma^\mu A_\mu$$

$$\mathrm{i}\,\mathcal{M}_2=\bar{u}_{s'}(p')\,\left(-\mathrm{i}\,e\,\gamma^\nu\right)\frac{\mathrm{i}\big(\not{p}-\not{k}+m\big)}{(p-k')^2-m^2}\left(-\mathrm{i}\,e\,\gamma^\mu\right)u_s(p)\,\varepsilon_{\nu\lambda}(k)\,\varepsilon^*_{\mu\lambda'}(k')$$

Now it is just a matter of turning the crank.

$$(p+k)^2-m^2=\underbrace{p^2}_{=m^2}+2p\cdot k+\underbrace{k^2}_{\substack{=0\\ \text{for}\\ \text{photons}}}-m^2=2\,p\cdot k$$

$$(p-k')^2-m^2=-\,2\,p\cdot k'$$

$\big(\not{p}+m\big)\gamma_\mu u_s(p)=-\,\gamma_\mu\big(\not{p}-m\big)\,u_s(p)+\big\{\gamma_\mu,\not{p}\big\}u_s(p)$. The first term is zero, due to the Dirac equation. $\big(\not{p}+m\big)\gamma_\mu u_s(p)=\big\{\gamma_\mu,\not{p}\big\}u_s(p)=2p_\mu\,u_s(p)$.

$$\mathrm{i}\,\mathcal{M}=\mathrm{i}\,\varepsilon^2\,\bar{u}_{s'}(p')\left(\frac{\gamma^\mu\not{k}\,\gamma^\nu+2\,p^\nu\gamma^\mu}{2\,p\cdot k}+\frac{\gamma^\nu\not{k}'\,\gamma^\mu-2\,\gamma^\nu p^\mu}{2\,p\cdot k'}\right)u^s\,\varepsilon_{\nu\lambda}\,\varepsilon^*_{\mu\lambda'}$$

We want to shoot unpolarised light at the electron:

$$\frac{1}{4}\sum_{\substack{\lambda,\lambda'\\ s,s'}}|\mathcal{M}|^2=$$

Fermi part requires taking

$$\sum_{s,s'}\big|\bar{u}_{s'}(p')\,(\;\not{}\;)\,u_s(p)\big|^2=\sum_{s,s'}\bar{u}_{s'}(p')\,(\;\not{}\;)u_s(p)\,\bar{u}_s(p)\,\gamma_0\,(\;\not{}\;)^\dagger\gamma_0\,u_{s'}(p')$$

$$\gamma_0\gamma^{\mu\dagger}\gamma_0=\gamma^\mu$$

$$\gamma_0(\gamma^\mu\gamma^\nu\gamma^\sigma)^\dagger\gamma_0=\gamma^\sigma\gamma^\nu\gamma^\mu$$

$$\sum_{s,s'}\big|\bar{u}_{s'}(p')\,(\;\not{}\;)\,u_s(p)\big|^2=\sum_{s,s'}\bar{u}_{s'}(p')\,(\;\not{}\;)\underbrace{u_s(p)\,\bar{u}_s(p)}_{=\not{p}+m}\gamma_0\,(\;\not{}\;)^\dagger\gamma_0\,u_{s'}(p')=$$

$$=\mathrm{tr}\big(\not{p}'+m\big)\,(\;\not{}\;)^{\mu\nu}\,\big(\not{p}+m\big)\,(\;\not{}\;)^{\rho\sigma}$$

I'm making a mess. Let's back-track.

$$\sum_{\substack{s,s'\\ \lambda,\lambda'}} \left|\bar{u}_{s'}(p')\,(\ \not\ \ )^{\mu\nu}\,u_s(p)\varepsilon_\mu^*\varepsilon_\nu\right|^2 =$$

$$\sum_{\substack{s,s'\\ \lambda,\lambda'}} \bar{u}_{s'}(p')\,(\ \not\ \ )^{\mu\nu}\underbrace{u_s(p)\,\bar{u}_s(p)}_{=\not p+m}\gamma_0\,(\ \not\ \ )^{\rho\sigma\dagger}\gamma_0\,u_{s'}(p')\,\varepsilon_{\mu\lambda}^*\varepsilon_{\nu\lambda'}\,\varepsilon_{\rho\lambda}\,\varepsilon_{\sigma\lambda'}^* =$$

$$= \underbrace{\mathrm{tr}(\not p'+m)(\ \not\ \ )^{\mu\nu}\,(\not p+m)\,(\ \not\ \ )^{\rho\sigma}}_{\substack{\text{trace of }\gamma\text{ matrices}\\ \text{same techniques as we did last time}}}\underbrace{\sum_\lambda \varepsilon_{\mu\lambda}^*\varepsilon_{\rho\lambda}\sum_{\lambda'}\varepsilon_{\nu\lambda'}\,\varepsilon_{\sigma\lambda'}^*}_{?}$$

$$\sum_{\lambda=1,2}\varepsilon_{\mu\lambda}^*\varepsilon_{\rho\lambda}, \quad \varepsilon_{\mu,\lambda} = \begin{pmatrix} 0 \\ \boldsymbol{\varepsilon}_\lambda \end{pmatrix}, \quad \boldsymbol{\varepsilon}_\lambda\cdot\boldsymbol{k} = 0$$

It does not look too good.

$\varepsilon_\mu \to \varepsilon_\mu + \alpha\,k_\mu$ is a gauge transformation.

$$k\cdot\varepsilon = 0, \quad k\cdot(\varepsilon+\alpha k) = 0 + \alpha k^2 = \alpha\cdot 0 = 0$$

$$N_{\nu\rho} = \sum_{\lambda=1,2}\varepsilon_{\mu\lambda}\,\varepsilon_{\rho\lambda}^* = ?$$

N must be transverse: $k^\mu\,N_{\mu\rho} = k^\rho N_{\mu\rho} = 0$. This is not a unique quantity. Introduce an arbitrary vector $\hat{l}_\mu$ which is lightlike ($\hat{l}^2 = 0$), with $\hat{l}\cdot k \neq 0$.

$$N_{\mu\rho} = -\,g_{\mu\rho} + \frac{\hat{l}_\mu\,k_\rho + \hat{l}_\rho k_\mu}{\hat{l}\cdot k}$$

But when contracted with the fermionic part, the bad stuff disappears.

$$\sum_{\lambda=1,2}\varepsilon_{\mu\lambda}^*\varepsilon_{\rho\lambda} \mathrel{\hat{=}} -\,g_{\mu\rho}$$

It is not equal to $-\,g_{\mu\rho}$, but gauge stuff makes it possible to pretend it is.

$$\frac{1}{4}\sum_{\substack{\lambda,\lambda'\\ s,s'}} |\mathcal{M}|^2 = \frac{e^4}{4}\ \mathrm{tr}\big((\not p+m)(\ \not\ \ )^{\mu\nu}(\not p+m)(\ \not\ \ )_{\nu\mu}\big) =$$

$$= \frac{e^4}{4}\left(\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}+\mathrm{III}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right)$$

$$\mathrm{I} = \mathrm{tr}\big((\not p'+m)(\gamma^\mu\not k\gamma^\nu + 2\gamma^\nu p^\mu)(\not p+m)(\gamma_\nu\not k\gamma_\mu + 2\,\gamma_\mu p_\nu)\big)$$

$$\gamma^\nu\not p\gamma_\nu = p_\mu\,\gamma^\nu\gamma^\mu\gamma_\nu = p_\mu(\{\gamma^\nu,\gamma^\mu\}\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_\nu) =$$

$$[\gamma^\nu\gamma_\nu = \gamma_0^2 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 = 1 - (-1) - (-1) - (-1) = 4] =$$

$$= p_\mu(2g^{\mu\nu} - 4\gamma^\mu) = -2p_\mu\gamma^\mu = -2\not p$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right)$$

Kinematics

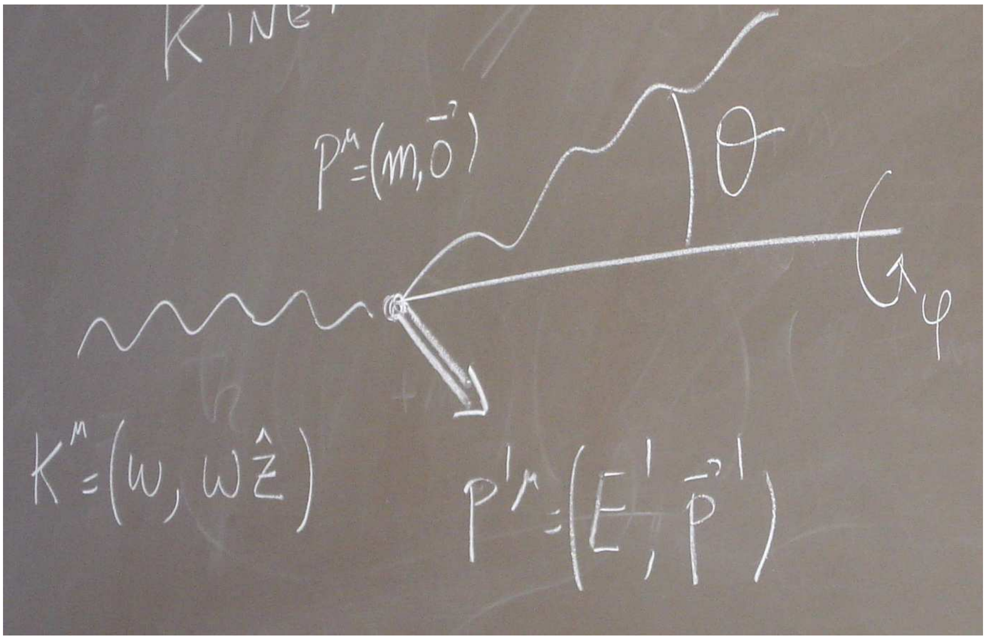

Figure 3.

$$k'^{\mu}(\omega', \omega'\sin\theta\,\hat{\boldsymbol{x}}+\omega'\cos\theta\,\hat{\boldsymbol{z}})$$

$$p'^2=m^2=(p+k-k')^2=p^2+k^2+k'^2\,p\cdot(k-k')-2\,k\cdot k'=$$

$$=m^2+0+0+2\,m(\omega-\omega')-2\,\omega\,\omega'(1-\cos\theta)$$

$$m(\omega-\omega')-\omega\,\omega'(1-\cos\theta)=0 \quad \Rightarrow \frac{\Delta\lambda}{2\pi}=\frac{1}{\omega'}-\frac{1}{\omega}=\frac{1-\cos\theta}{m}$$

This is what is usually called the Compton formula.

$$\mathrm{d}\sigma=\frac{1}{2\,E_{e^-}\,2\,E_{\gamma}\,|v_e-v_{\gamma}|}\cdot\frac{1}{4}|\mathcal{M}|^2\,\mathrm{d}\,\mathrm{Lips}_2=\frac{1}{2m\cdot 2\,\omega\cdot 1}\cdot 2\varepsilon^4\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right)\mathrm{d}\,\mathrm{Lips}_2$$

$$\mathrm{dLips}_2=\frac{\mathrm{d}^3k'}{(2\pi)^3}\,\frac{1}{2E_{k'}}\,\frac{\mathrm{d}^3p'}{(2\pi)^3}\,\frac{1}{2E_{p'}}\,(2\pi)^4\delta^4(k'+p'-k-p)$$

$$=\frac{1}{16\pi^2}\,\frac{\mathrm{d}^3\,k'}{\omega'\,E'}\,\delta^{(1)}(\omega'+E'-\omega-m)$$

$$\left[E'=\sqrt{m^2+p'^2}=\sqrt{m^2+\omega^2+\omega'^2-2\,\omega\,\omega'\cos\theta}\right]$$

$$=\frac{1}{16\pi^2}\int\frac{\omega'^2\,\mathrm{d}\omega'\cdot 2\pi\,\mathrm{d}\cos\theta}{\omega'\cdot E'}\,\delta(\omega'+E'-\omega-m)=\frac{1}{8\pi}\cdot\frac{\omega'}{E'}\int\mathrm{d}\cos\theta\cdot\frac{1}{|1+\frac{\omega'-\omega\cos\theta}{E'}|}=$$

$$=\frac{1}{8\pi}\int\mathrm{d}\cos\theta\cdot\frac{\omega'}{|E'+\omega'-\omega\cos\theta|}=\frac{1}{8\pi'}\int\mathrm{d}\cos\theta\,\frac{\omega'^2}{\omega\,m}$$

$$\mathrm{d}\sigma=\ldots\,\mathrm{d}\cos\theta$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}=\frac{1}{2\omega}\,\frac{1}{2m}\cdot\frac{1}{8\pi}\,\frac{\omega'^2}{\omega\,m}\cdot\frac{1}{4}\sum|\mathcal{M}|^2=\frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'(\theta)}{\omega}+\frac{\omega}{\omega'(\theta)}-\sin^2\theta\right)$$

$$\text{where}\,\alpha=\frac{e^2}{4\pi}.$$

This is the Klein–Nishina equation. $\omega' = \omega'(\omega, \theta)$.

Low energy limit: $\omega' \approx \omega$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(2 - \sin^2\theta\right)$$

$$\sigma_T = \int_{-1}^{+1} \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{8\pi\alpha^2}{3\, m}$$