2009–02–18

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Finally we can calculate a process that exists, that we can go to the lab and measure. A very important process: $e^+ + e^- \to \mu^+ + \mu^-$. The muon is like a heavy electron: $m_e \approx 0.5\,\text{MeV}$, $m_\mu \approx 105\,\text{MeV}$. There is only one Feynman diagram that contributes:

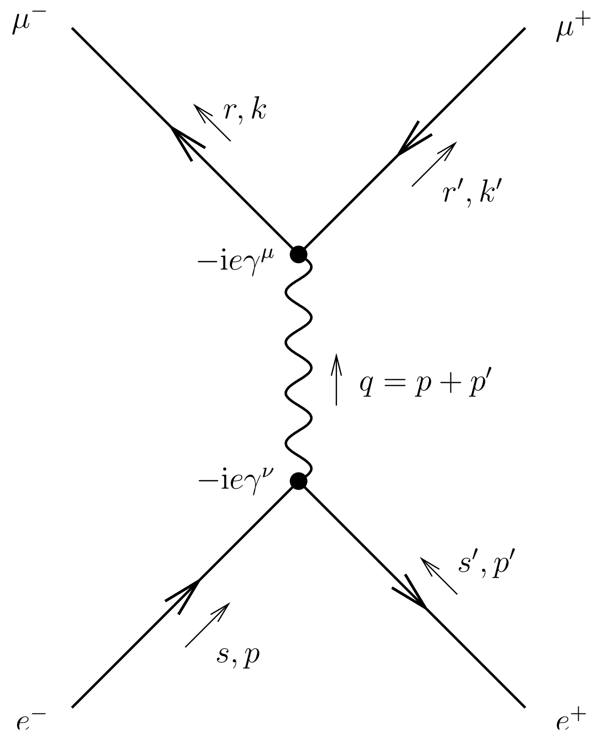

Figure 1. $e^+ + e^- \to \mu^+ + \mu^-$

In the centre of mass frame,

$$P^\mu = (E_e, \boldsymbol{p} = p\hat{\boldsymbol{z}})$$

$$P'^\mu = (E_e, -\boldsymbol{p})$$

$$E_{\text{cm}} = 2\,E_e \equiv \sqrt{s} \geqslant 2\,m_\mu \ \text{ (otherwise the process is not possible)}$$

$E = \gamma\, m, P = \beta\,\gamma\, m$. $E_e = \gamma_e\, m_e > m_\mu$

$$\Rightarrow \quad \gamma_e > \frac{m_\mu}{m_e} \approx 200$$

Remember $\gamma = 1/\sqrt{1-\beta^2}$, with β being the velocity. ($c = 1$).

The mass of the electron is negligible in this case, so that we can take $E_e = |\boldsymbol{p}|$, instead of $E_e = \sqrt{|\boldsymbol{p}|^2 + m_e^2}$. The muons, on the other hand, could be very well be non-relativistic.

$$\mathrm{i}\,\mathcal{M} = \bar{u}_r(k)(-\mathrm{i}e\,\gamma^\mu)\,v_{r'}(k')\left(\frac{-\mathrm{i}\,g_{\mu\nu}}{q^2}\right)\bar{v}_s(p')\,(-\mathrm{i}\,e\,\gamma^\nu)\,u_s(p)$$

This is called the transition amplitude.

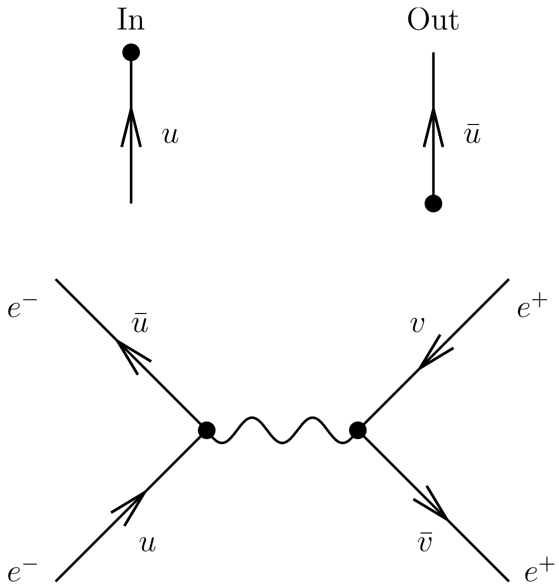

Figure 2. Feynman diagrams for fermions.

$$\frac{\mathrm{d}\sigma_{\mathrm{pol}}}{\mathrm{d}\Omega} = \frac{|\boldsymbol{k}|}{32\,\pi^2\,E_{\mathrm{cm}}^3}\,|\mathcal{M}|^2$$

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p\cdot p')^2 - \cancel{m_e^4}}}\cdot(2\pi)^4\,\delta^{(4)}(p+p'-k-k')\,\frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^3\,2E_k\,(2\pi)^3\,2E_k'}$$

(The denominator is something of a flux.)

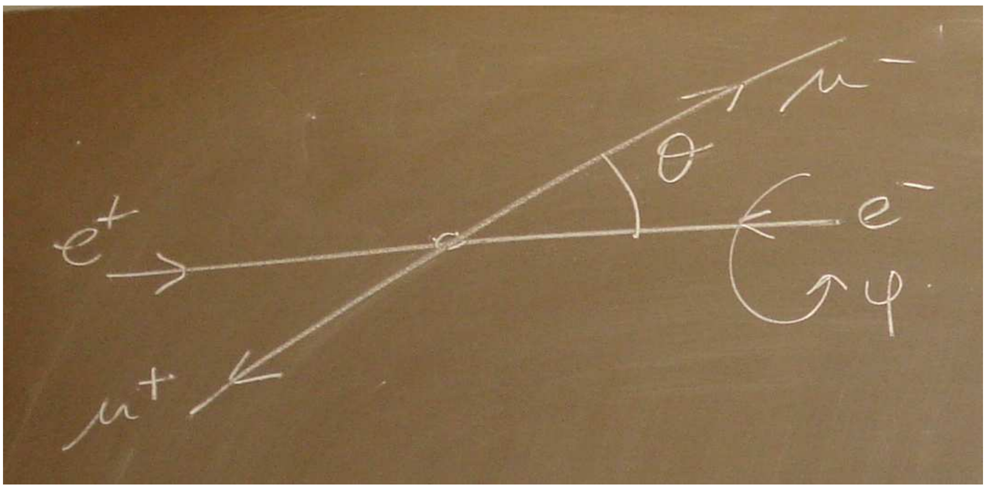

Figure 3. $\mathrm{d}\Omega = \mathrm{d}(\cos\theta)\,\mathrm{d}\varphi$

Let us look at

$$\frac{\mathrm{d}\sigma_{\mathrm{pol}}}{\mathrm{d}\Omega} = \frac{|\boldsymbol{k}|}{32\,\pi^2\,E_{\mathrm{cm}}^3}\,|\mathcal{M}|^2.$$

$$|\bar{u}\,\gamma^\mu\,v|^2 = \bar{u}\,\gamma^\mu v\cdot(\bar{u}\,\gamma_\mu\,v)^*$$

Important trick: $(\bar{u}\,\gamma_\mu v)^* = (\bar{u}\,\gamma_\mu\, v)^\dagger = \left(u^\dagger\,\gamma_0\,\gamma_\mu\, v\right)^\dagger = v^\dagger\gamma_\mu^\dagger\gamma_0^\dagger u = v^\dagger\gamma_\mu^\dagger\gamma_0\, u$. Insert $\gamma_0\gamma_0 = 1$. $(\bar{u}\,\gamma_\mu v)^* = \bar{v}\gamma_0\gamma_\mu^\dagger\gamma_0\, u = \bar{v}\,\gamma_\mu\, u$.

$$|\bar{u}\,\gamma^\mu\, v|^2 = \bar{u}\,\gamma^\mu v\cdot(\bar{u}\,\gamma_\mu\, v)^* = \bar{u}\gamma^\mu\, v\,\bar{v}\,\gamma_\mu\, u$$

We will not continue to write down the momenta all the time, because they are always associated with the spin: we had s with p, s' with p', r with k and r' with k'.

$$|\mathcal{M}|^2 = \frac{e^4}{(q^2)^2}\underbrace{(\bar{u}_r\gamma^\mu v_{r'})(\,\bar{v}_{s'}\,\gamma_\mu\, u_s)}_{\mathcal{M}}\cdot\underbrace{(\bar{v}_{r'}\gamma^\nu\, u_r)\,(\bar{u}_s\,\gamma_\nu\, v_{s'})}_{\mathcal{M}^*} =$$

$$= \frac{e^4}{(q^2)^2}(\bar{u}_r\gamma^\mu v_{r'})(\bar{v}_{r'}\gamma^\nu u_r)(\bar{v}_{s'}\,\gamma_\mu\, u_s)(\bar{u}_s\gamma_\nu\, v_{s'})$$

This presumes that you can control the incoming polarisation (spin up/spin down) and measure the polarisation of the muons. In practise you don't do this. You use unpolarised stuff.

$$\frac{\mathrm{d}s_{\mathrm{unp}}}{\mathrm{d}\Omega} =$$

Average the incoming spins and sum the outgoing spins. A muon of spin up will leave the same track as a muon of spin down.

$$\frac{\mathrm{d}s_{\mathrm{unp}}}{\mathrm{d}\Omega} = \frac{1}{2}\cdot\frac{1}{2}\sum_{s,s',r,r'}\frac{\mathrm{d}\sigma_{\mathrm{pol}}}{\mathrm{d}\Omega}$$

$$\sum_{r r'\, s s'}(\bar{u}_r\gamma^\mu v_{r'})(\bar{v}_{r'}\gamma^\nu u_r)(\bar{v}_{s'}\,\gamma_\mu\, u_s)(\bar{u}_s\gamma_\nu\, v_{s'})$$

Use

$$\sum_s u^s(p)\,\bar{u}^{\,s}(p) = \not{p} + m_e \approx \not{p}, \qquad \text{matrices: } \underbrace{\begin{pmatrix} \\ \vdots \\ \\ \end{pmatrix}}_{=u^s(p)}\underbrace{\begin{pmatrix} & \cdots & \end{pmatrix}}_{\bar{u}^s(p)} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\sum_{r'} v^{r'}(k')\,\bar{v}^{\,r'}(k') = \not{k} - m_\mu$$

$$\sum_{r r'}(\bar{u}_r\gamma^\mu v_{r'})(\bar{v}_{r'}\gamma^\nu u_r)\sum_{s s'}(\bar{v}_{s'}\,\gamma_\mu\, u_s)(\bar{u}_s\gamma_\nu\, v_{s'})$$

$$= \sum_r \bar{u}_r\gamma^\mu\left(\not{k}' - m_\mu\right)\gamma^\nu\, u_r\sum_{s'}\bar{v}_{s'}\gamma_\mu\not{p}\gamma_\nu v_{s'} =$$

$$= \mathrm{tr}\left(\not{k} - m_\mu\right)\gamma^\mu\left(\not{k}' - m_\mu\right)\gamma^\nu\cdot\mathrm{tr}\left(\not{p}'\,\gamma_\mu\not{p}\;\gamma_\nu\right)$$

$$\sum_{a b}\sum_{r=\pm\frac{1}{2}}\bar{u}_{(r)}^a\, X_a^b\, u_{(r)\,b}, \quad a,b = 1,2,3,4, \quad u = \begin{pmatrix} \\ \vdots \\ \\ \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} & \cdots & \end{pmatrix}$$

$$\sum_{r=\pm\frac{1}{2}}\bar{u}_{(r)}^a\, X_a^b\, u_{(r)\,b} = \sum_{a,b}\underbrace{\sum_{r=\pm\frac{1}{2}} u_{(r)b}\,\bar{u}_{(r)}^a}_{\not{p}_b^a} = \sum_{a,b}\not{p}_b^a\, X_a^b = \mathrm{tr}\,\not{p}X$$

"Math is so simple they shouldn't really be teaching it."

What is $\mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda)$? This goes under the pompous name of "trace formulae".

$$\mathrm{tr}(1) = \mathrm{tr}(\mathbb{I}_{4\times 4}) = 4, \quad \mathrm{tr}\Big(\big(\gamma^0\big)^2 \Big) = \mathrm{tr}(1)$$

$$\mathrm{tr}(\gamma^\mu) = 0$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu) = \frac{1}{2}\,\mathrm{tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \frac{1}{2}\,\mathrm{tr}(2\, g^{\mu\nu}\,\mathbb{I}) = g^{\mu\nu}\,\mathrm{tr}\,\mathbb{I} = 4\, g^{\mu\nu}$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0$$

$\big(\gamma^5\big)^2 = 1.$

$$\mathrm{tr}\big(\gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\big) = -\,\mathrm{tr}\big(\gamma^5 \gamma^\mu\, \gamma^\nu\, \gamma^\rho\, \gamma^5\big) = -\,\mathrm{tr}\big(\gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\big)$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu\, \gamma^\rho\, \gamma^\tau) = 4\,(g^{\mu\nu}\, g^{\rho\tau} - g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau}\, g^{\nu\rho})$$

$$\mathrm{tr}\big(\not{p}'\, \gamma^\mu \not{p}\, \gamma^\nu\big) = P'_\rho\, P_\sigma\, \mathrm{tr}\,(\gamma^\rho \gamma^\mu \gamma^\sigma\, \gamma^\nu) = 4\big(P'^\mu P^\nu + P'^\nu P^\mu - P' \cdot P g^{\mu\nu}\big)$$

$$\mathrm{tr}\big(\not{k} - m_\mu\big)\gamma^\mu\big(\not{k} - m_\mu\big)\gamma^\nu = \mathrm{same} + 4\, m_\mu^2\, g^{\mu\nu}$$

$$\frac{1}{4}\sum_{s s' r r'} |\mathcal{M}|^2 = \frac{8\, e^4}{q^4}\big((p\cdot k)(p'\cdot k') + (p\, k')(p'\, k) + m_\mu^2\, p\, p'\big)$$

$P'^\mu = (E, -\,E\,\hat{\boldsymbol{z}})$

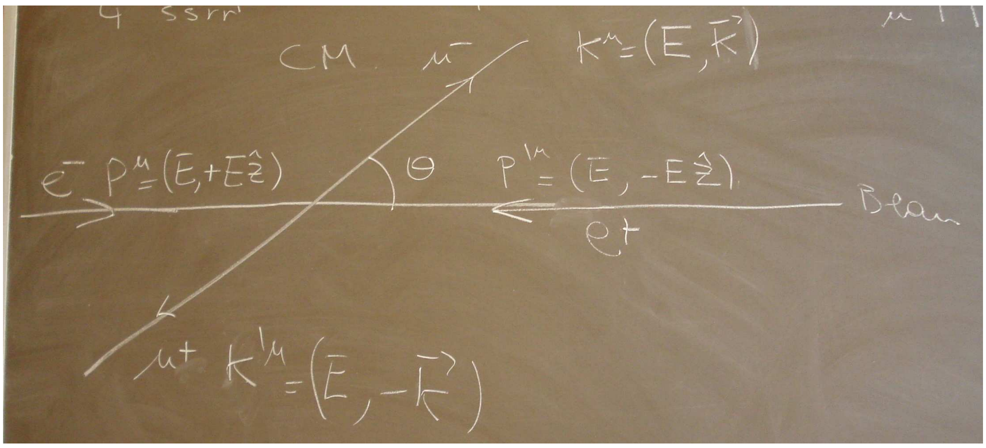

Figure 4.

$E = \sqrt{\boldsymbol{k}^2 + m_\mu^2}.$

$\boldsymbol{k}\cdot\hat{\boldsymbol{z}} = |\boldsymbol{k}|\cos\theta = \sqrt{E^2 - m_\mu^2\cos\theta}.$

$$p\cdot k = p^0\,k^0 - \boldsymbol{p}\cdot\boldsymbol{k} = E^2 - E\,|\boldsymbol{k}|\cos\theta$$

$$q^\mu = p^\mu + p'^\mu, \quad q^2 = (2E)^2 = E_{\mathrm{cm}}^2$$

$$\frac{1}{4}\sum_{\mathrm{spins}} |\mathcal{M}|^2 = \frac{8\,e^4}{16E^4}\left(\left(E^2 - E|\boldsymbol{k}|\cos\theta\right)(\cdots) + \cdots\right) = e^4\left(\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

First physics result of the course:

$$\left.\frac{\mathrm{d}\sigma_{\mathrm{unpol}}}{\mathrm{d}\Omega}\right|_{\substack{e^+e^-\to\mu^+\mu^-\\ \text{centre of mass}}} = \frac{|\boldsymbol{k}|}{32\,\pi^2\,E_{\mathrm{cm}}^3}\cdot\frac{1}{4}\sum|\mathcal{M}|^2$$

Fine structure constant: $\alpha = e^2/4\pi$.

$$\left.\frac{\mathrm{d}\sigma_{\mathrm{unpol}}}{\mathrm{d}\Omega}\right|_{\substack{e^+e^-\to\mu^+\mu^-\\ \text{centre of mass}}} = \frac{\alpha^2}{16\,E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

$$\sigma_{\mathrm{total}} = \int \mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi\alpha^2}{3\,E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\,\frac{m_\mu^2}{E^2}\right)$$

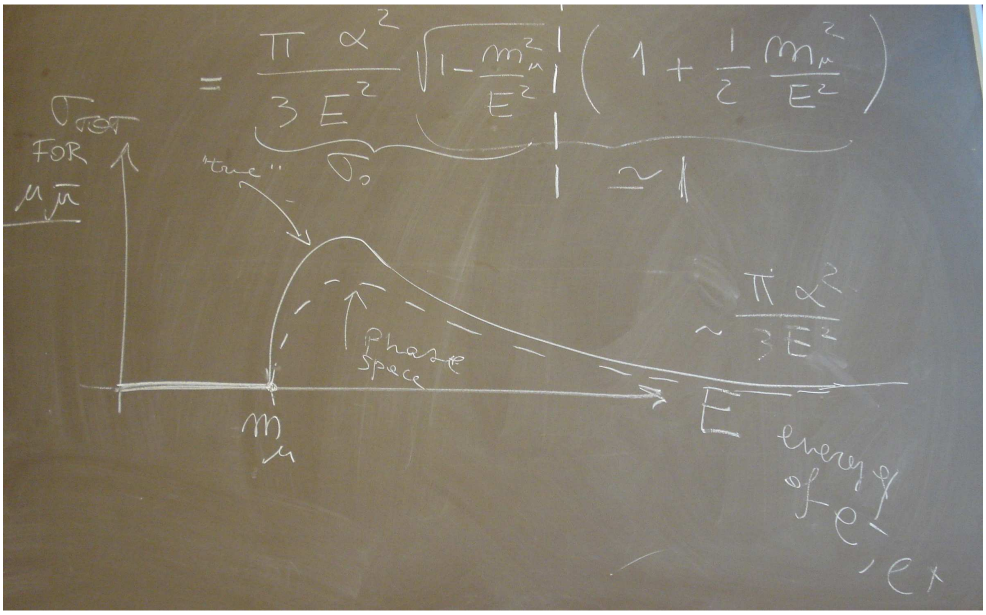

Figure 5. σ_{tot} versus E

$$\sigma_{\mathrm{tot}} = \underbrace{\frac{\pi\alpha^2}{3\,E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}}_{\sigma_0}\left(1+\frac{1}{2}\,\frac{m_\mu^2}{E^2}\right)$$

σ_0 is the phase space contribution.

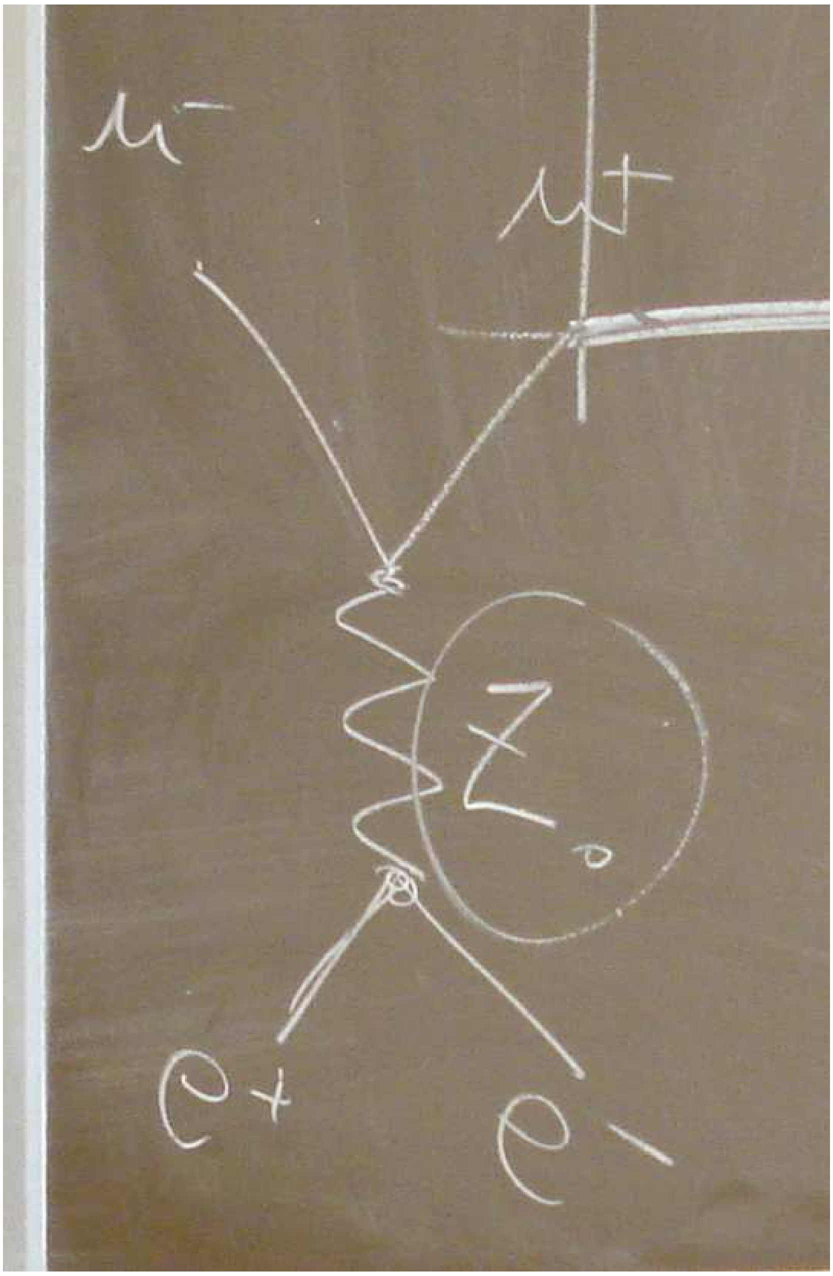

Figure 6.