2009–02–06

Lecturer: Per Salomonson

This is the last lecture about chapter 4 in Peskin&Schroeder. Also: Maxwell field, Renormalisability.

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^{\nu} A_{\nu}$$

We could use operators as before, but quantising vector fields is more easily done using path integrals.

Equations of motion:

$$\partial^2 A^{\nu} - \partial^{\nu} \partial \cdot A = j^{\nu}$$

Fourier transform this:

$$A^{\nu}(x) = (2\pi)^{-3} \int \mathrm{d}^4 p \, \tilde{A}^{\nu}(p) \, \mathrm{e}^{-\mathrm{i}p \cdot x}$$

$$\Rightarrow \underbrace{\left(p^2 \delta^{\nu}{}_{\mu} - p^{\nu} p_{\mu}\right)}_{\substack{4\times 4 \text{ matrix has} \\ \text{eigenvalues} \\ (p^2, p^2, p^2, 0)}} \tilde{A}^{\mu} = -\tilde{j}^{\nu}$$

Multiply by p_{ν}:

$$0 = -p_{\nu} \tilde{j}^{\nu}$$

Require $p_{\nu} j^{\nu} = 0 \Leftrightarrow$ conserved current.

The 4×4 matrix is not invertible $\Rightarrow$ propagator cannot be constructed the usual way. For now, use quick fix. Note that the photon field A^{μ} has 2 degrees of freedom, complete set of polarisation vectors $\varepsilon_s^{\mu}(\pi)$, $s = 1, 2$.

$$\varepsilon_s^0 = 0, \quad \varepsilon_s^{\mu} p_{\mu} = 0$$

also orthogonality $\varepsilon_s^{\mu}(\boldsymbol{p}) \, s_{s'\,\mu}^{*}(\boldsymbol{p}) = -\delta_{s s'}$.

Ansatz for A^{μ}: $E_p = p^0 = |\boldsymbol{p}|$.

$$A^{\mu}(x)_{\mathrm{op}} = (2\pi)^3 \int \frac{\mathrm{d}^3 p}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_s(\boldsymbol{p}) \, \varepsilon_s^{\mu}(\boldsymbol{p}) \, \mathrm{e}^{-\mathrm{i}p \cdot x} + a_s^{\dagger}(\boldsymbol{p}) \, \varepsilon_s^{\mu *}(\boldsymbol{p}) \, \mathrm{e}^{\mathrm{i}p \cdot x} \right)$$

$$[a_s(\boldsymbol{p}), a_{s'}(\boldsymbol{p}')] = (2\pi)^3 \, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \, \delta_{s s'}$$

$$\langle 0 | T A^{\mu}(x) \, A^{\nu}(x) \, | 0 \rangle = \langle 0 | \theta(x^0 - y^0) \, A_+^{\mu}(x) \, A_-^{\nu}(y) + \theta(y^0 - x^0) \, A_+^{\nu}(\psi) \, A_-^{\mu}(x) | 0 \rangle =$$

$$= (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p} \sum_s \left(\theta\left(x^0 - y^0\right) \varepsilon_s^{\mu} \, \varepsilon_s^{\nu *} \, \mathrm{e}^{-\mathrm{i}p \cdot (x - y)} + \theta\left(y^0 - x^0\right) \varepsilon_s^{\nu}(p) \, \varepsilon_s^{\mu *}(p) \, \mathrm{e}^{\mathrm{i}p \cdot (x - y)} \right)$$

If e.g. $p^{\mu} = (|\boldsymbol{p}|, |\boldsymbol{p}|, 0, 0)$: $\boldsymbol{p}$ points in the positive x direction.

$$\sum_s \varepsilon_s^{\mu} \, \varepsilon_s^{\nu *} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\mu\nu} = -\left(\eta^{\mu\nu} - \frac{p^{\mu} \tilde{p}^{\nu} + \tilde{p}^{\mu} p^{\nu}}{2|\boldsymbol{p}|^2} \right)$$

with $p^\mu = (|\boldsymbol{p}|, \boldsymbol{p})$ and $\tilde{p}^\mu = (|\boldsymbol{p}|, -\boldsymbol{p})$. We are doing Coulomb gauge quantisation. We drop the last term [a bit unclear why].

$$D_{\mu\nu}(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p} (-\eta^{\mu\nu}) \big(\theta(x\,0)\, \mathrm{e}^{-\mathrm{i}p\cdot x} + \theta(-x^0)\, \mathrm{e}^{\mathrm{i}p\cdot x}\big) =$$

$$= (2\pi)^{-4} \int \mathrm{d}^4 p(-\eta^{\mu\nu}) \frac{\mathrm{i}}{p^2 + \mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}p\cdot x}$$

Remark: In lecture 6, I defined

$$S_{qp} = \langle q_1, \ldots, q_n | U(T, 0) | p_1, p_2 \rangle$$

$$U(T, 0) = \mathrm{e}^{\mathrm{i}H_0 t}\, \mathrm{e}^{-\mathrm{i}Ht} = T \exp\bigg(-\mathrm{i} \int_0^t \mathrm{d}t'\, H_{II}(t')\bigg)$$

1) Note H_0 just contributes a phase.

2) S_{qp} = scattering matrix element of Peskin&Schroeder, except they use $\tilde{a}^\dagger(p)|0\rangle$. I used $a^\dagger(p)_{\mathrm{Box}}|0\rangle$. $\tilde{a}^\dagger(\boldsymbol{p}) = \sqrt{2E_p V/(2\pi)^3}\, a^\dagger(\boldsymbol{p})|_{\mathrm{Box}}$

$$(S_{qp})_{P\&S} = \mathbb{I} + (2\pi)^4\, \delta^4\Big(p_1 + p_2 - \sum q_i\Big)\, \mathrm{i}\mathcal{M}$$

Which interactions are possible in Quantum Field Theory. Renormalisability $\Rightarrow$ strong restrictions. To describe them, use dimensional analysis. We use units such that $\hbar = 1$ and $c = 1$:

$$L = T, \quad ET = 1 \quad \Rightarrow \quad \text{only one unit left}$$

Use mass $M = E = 1/T = 1/L$.

$$\bigg[\int \mathcal{L}\, \mathrm{d}^4 x\bigg] = \bigg[\int H\, \mathrm{d}t\bigg] = 1$$

$$[\mathcal{L}] = M^4$$

$$\mathcal{L}_0 = \frac{1}{2}\Big((\partial\varphi)^2 - m_\varphi^2 \varphi^2\Big) + \bar{\psi}\,(\mathrm{i}\partial\!\!\!/ - m_\psi)\,\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$[\partial] = M, \quad [\varphi] = M, \quad [A_\mu] = M, \quad [\psi] = M^{3/2}, \quad [m_\varphi] = [m_\psi] = M$$

In general $\mathcal{L}_I = \sum_k \lambda_k \mathcal{O}_k$.

Example:

$$\frac{1}{n!} \lambda_n\, \varphi^n \quad \Rightarrow \quad [\lambda_n] = M^{4-n}$$

$$e\, \bar{\psi} \gamma^\mu \psi\, A_\mu \quad \Rightarrow \quad [e] = M^{4 - 2\cdot\frac{3}{2} - 1} = M^0, \text{ dimensionless}$$

Dimensionality criterion:

If any coupling constant has mass dimension less than zero, then the theory is nonrenormalisable. λ_n has mass dimension $4 - n$.

Example: scalar φ^4 theory and quantum electrodynamics (QED) have mass dimension zero coupling constants. Such theories are just barely renormalisable $\Rightarrow$ they are specially interesting.

Example: Gravity. $E = M\,m\,G_{\mathrm{N}}/R \Rightarrow [G_{\mathrm{N}}] = [E\,R/M\,m] = M^{-2}$. Thus quantum gravity is not renormalisable.

Example:

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\not{\partial} - m\right)\psi - \gamma\left(\bar{\psi}\psi\right)^2$$

$$[\gamma] = M^{4-4\cdot\frac{3}{2}} = M^{-2}: \text{ nonrenormalisable.}$$

Problem arises as follows: consider two-particle scattering.

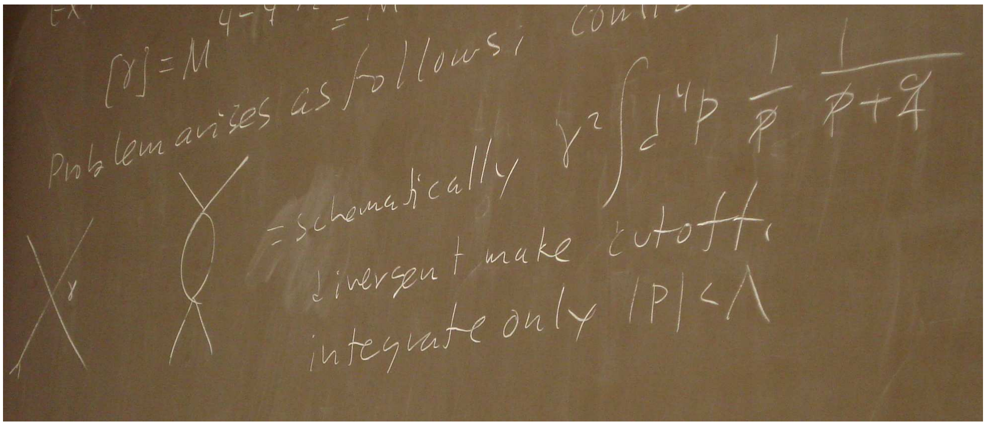

Figure 1.

schematically

$$\gamma^2 \int \mathrm{d}^4 p\, \frac{1}{\not{p}} \cdot \frac{1}{\not{p} + \not{q}} \sim \gamma^2\,\Lambda^2$$

divergent. Make cutoff. Integrate only $|p| < \Lambda$. In general $\times + \cdots = \gamma(1 + F(q, \Lambda, \gamma))$. $[F] = M^0$.

$$F(0, \Lambda, \gamma) = \sum_{n=1}^{\infty} c_n\, \gamma^n\, \Lambda^{2n}$$

Higher and higher powers of $\Lambda \Rightarrow$ nonrenormalisable.

Consider instead:

$$\mathcal{L} = \frac{1}{2}\Big((\partial\varphi)^2 - m^2\varphi^2\Big) - \frac{\lambda}{4!}\,\varphi^4$$

$$\left[m^2\right] = M^2, \quad [\lambda] = M^0: \quad \text{renormalisable}$$

$$\times\ + \ = \lambda + \lambda \int \frac{\mathrm{d}^4 p}{(p^2 + m^2)((p+q)^2 + m^2)} \sim \lambda + \lambda^2 \ln \Lambda$$

This divergence can be handled by redefining (renormalising) λ.

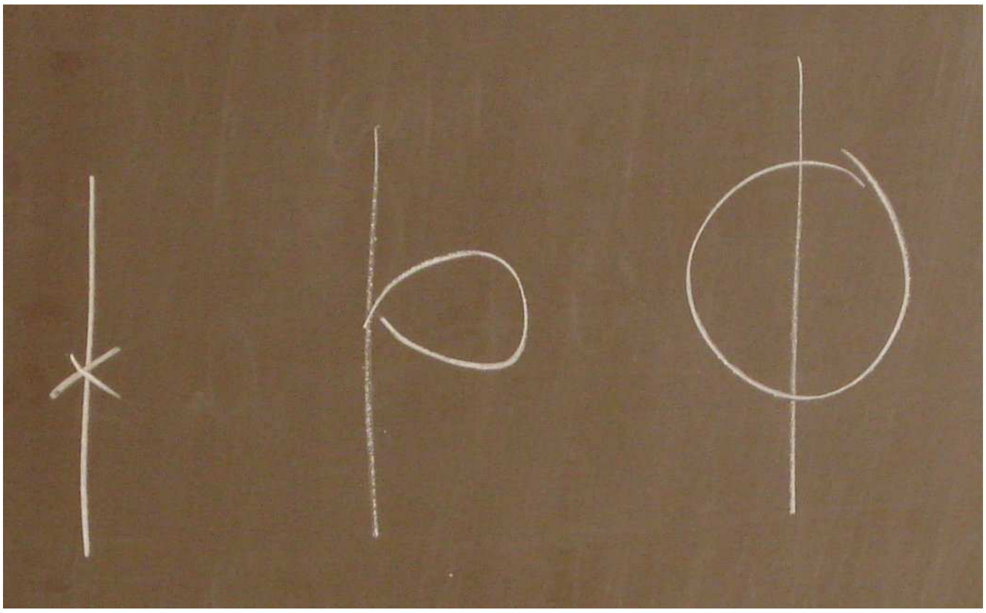

Figure 2. $m^2 + \lambda\,\Lambda^2 + \lambda^2\Lambda^2 + \cdots$. $F(0,\lambda,\Lambda)$ has dimension M^2. Can be $\sum c_n\lambda^n\Lambda^2$. Only quadratically divergent. Can be handled.

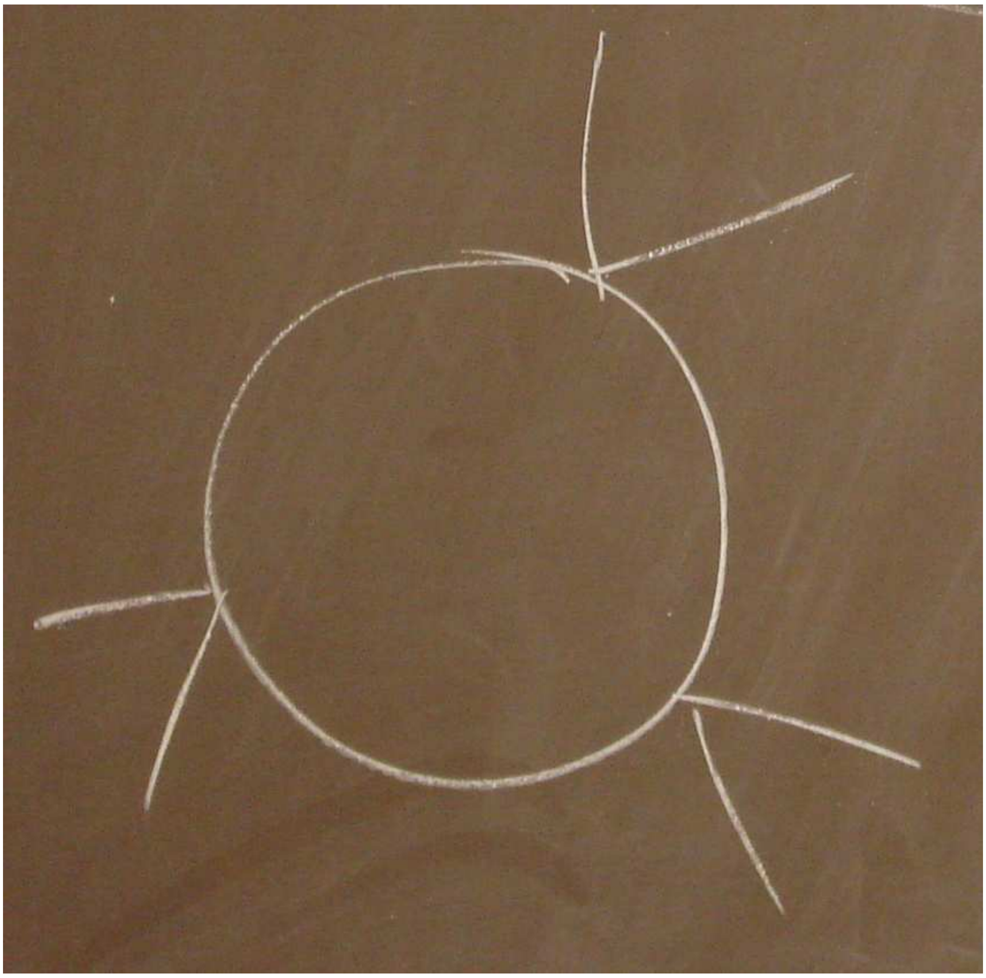

Figure 3. Has dimension M^{-2} $\Rightarrow$ convergent quantum corrections.

Example: Scalar QED = gauged $U(1)$ symmetry of a complex scalar field.

$$\mathcal{L}_0 = (\partial^\mu \varphi)^* \partial_\mu \varphi - m^2\, \varphi^* \varphi$$

U(1) symmetry. $\delta\varphi = -\,\mathrm{i}\varepsilon\varphi$. Gauging:

$$\partial_m \varphi \rightarrow \mathrm{D}_\mu \varphi = (\partial \mu + \mathrm{i}\varepsilon A_\mu)\varphi$$

$$\mathcal{L} = (\mathrm{D}_\mu \varphi)^* \mathrm{D}^\mu \varphi - m^2 \varphi^* \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} =$$

$$= \partial^\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \mathrm{i}\, \underbrace{e(\varphi^* \partial^\mu \varphi - \partial^\mu \varphi^* \varphi)}_{= j^\mu} A_\mu + e^2\ \varphi^* \varphi\, A^\mu A_\mu$$

$-\frac{\lambda}{4}(\varphi^* \varphi)^2$. $[\lambda] = M^0$. This term does not violate any symmetry, and should therefore be added to $\mathcal{L}$.

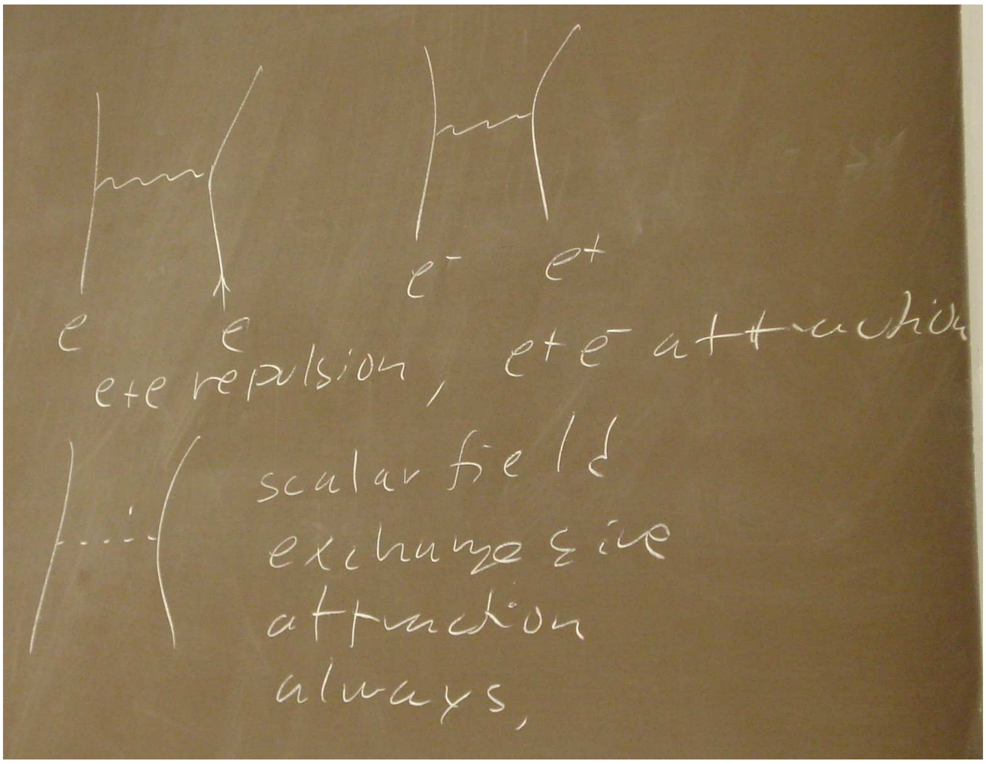

Figure 4. Scalar field exchange always gives attraction.