2009–02–04

Lecturer: Per Salomonson

Last time σ was calculated.

$$\mathcal{L}_I = -\frac{1}{4!}\,\lambda\,\varphi^4$$

$$p_1 + p_2 \to q_1 + q_2$$

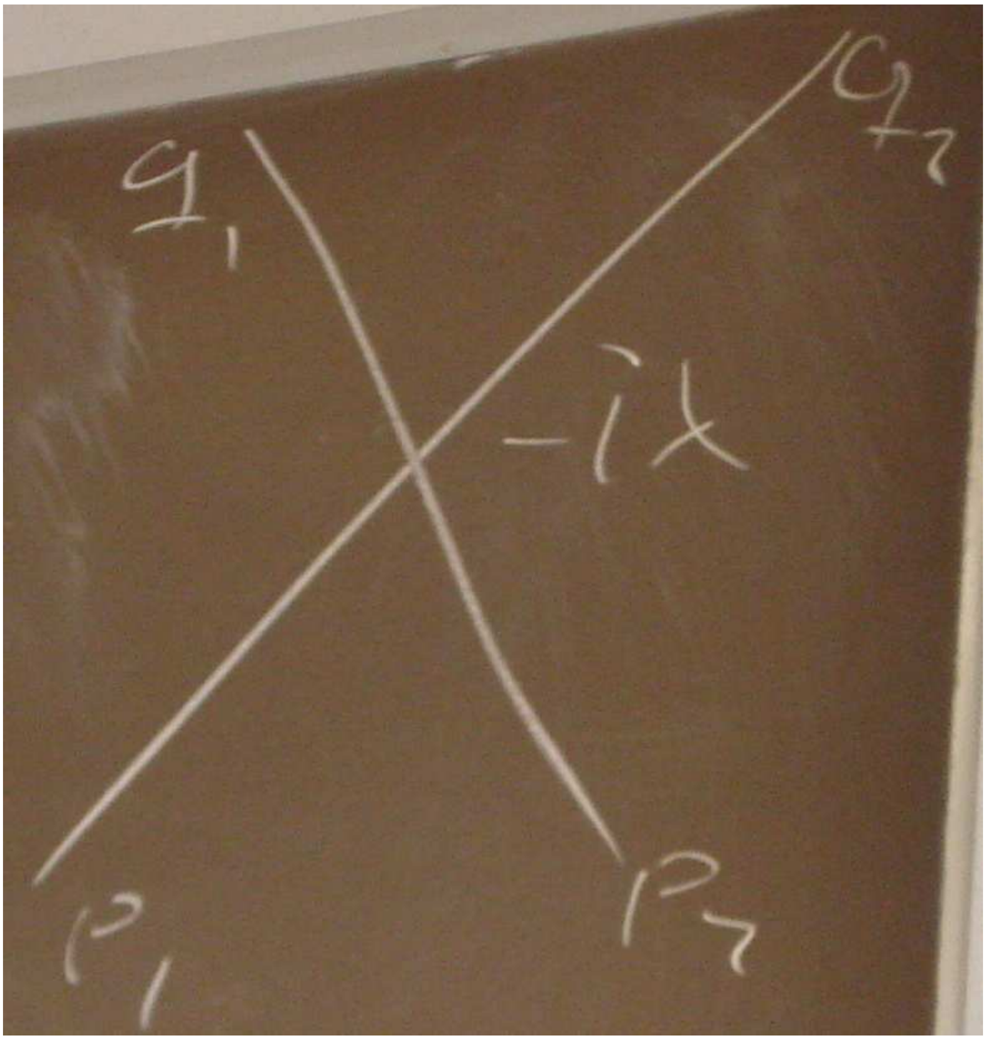

Figure 1. $p_1 + p_2 \to q_1 + q_2$.

Result, in the case $n = 2$, i.e. elastic scattering:

$$\mathrm{d}\sigma = \underbrace{\frac{1}{|\Delta v| \cdot 2E_{p_1} \cdot 2E_{p_2}}}_{\substack{\text{incoming particles}\\ \text{definition of } \sigma}} \underbrace{\prod_{i=1}^{2} \frac{\mathrm{d}^3 q_i}{(2\pi)^3 \cdot 2E_{q_i}}\,(2\pi)^4\,\delta^4\!\left(p_1 + p_2 - \sum_i q_i\right)}_{\substack{\text{dLips}\\ \text{differential Lorentz invariant phase space}}} \underbrace{|\mathrm{i}\mathcal{M}|^2}_{\substack{\text{“invariant}\\ \text{matrix}\\ \text{element”}}}$$

$$\mathrm{i}\,\mathcal{M} = -\,\mathrm{i}\lambda$$

$$\Delta v\, E_1\, E_2 = \left|\frac{p_2}{E_2} - \frac{p_1}{E_1}\right| =$$

(assume $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ directed in the z direction)

$$= |p_2^z E_1 - p_1^z E_1| = |\varepsilon_{\mu x y \nu} p_1^\mu p_2^\nu|$$

is invariant under boosts in the z direction. To get another σ in another theory, use the same procedure, get the same result, except $\mathrm{i}\mathcal{M}$ is different.

Example: $p_1 + p_2 \to q_1 + q_2$ in $\mathcal{L}_I = -\frac{1}{6}\lambda\varphi^3$. Then one needs two vertices, there are three Feynman graphs.

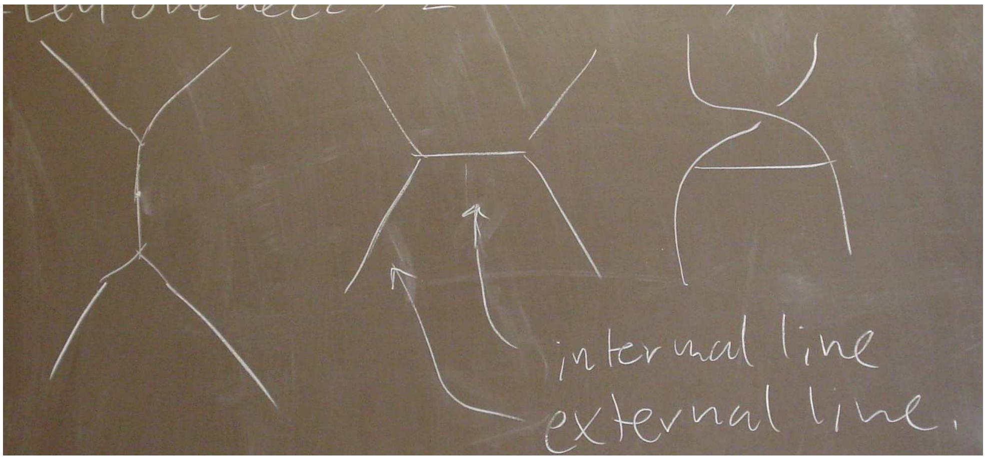

Figure 2. Two vertices give three graphs.

Now we have two vertices, so we must go to second order in the expansion

$$U(T,0) = T\exp\left(-\mathrm{i}\int \mathcal{H}_{II}\, \mathrm{d}^4 x\right)$$

$$-\frac{\mathrm{i}\lambda}{4!}\int \mathrm{d}^4 x\, \varphi^4(x) \to T\frac{1}{2}\left(-\frac{\mathrm{i}x}{6}\int \mathrm{d}^4 x\, \varphi^3(x)\right)\frac{-\mathrm{i}}{6}\lambda\int \mathrm{d}^4 y\, \varphi^3(y)$$

External line creation and annihilation operators eliminate four factors φ. Left is

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\cdot\frac{\mathrm{i}}{p^2 - m^2}\exp(-\mathrm{i}p(x-y))$$

Result

$$-\mathrm{i}\lambda \to (-\mathrm{i}\lambda)^2\left(\frac{\mathrm{i}}{(p_1+p_2)^2 - m^2} + \frac{\mathrm{i}}{(p_1-q_1)^2 - m^2} + \frac{\mathrm{i}}{(p_1-q_2)^2 - m^2}\right)$$

Feynman rules for computing $\mathrm{i}\mathcal{M}$ in sclar field theories:

$\mathrm{i}\mathcal{M}$ = sum of all connected amputated Feynman graphs evaluated as follows:

1. For each internal line with a momentum p, there is a factor $\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$.

2. For each external line, a factor 1.

3. For each vertex $\times$ we get $-\mathrm{i}\lambda$ if $\mathcal{L}_I = -\frac{\lambda}{4!}\varphi^4$.

4. Momentum conservation at each vertex.

5. Integrate $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ for each undetermined momentum.

6. Divide by symmetry factor.

These are the momentum space Feynman rules. There are also the position space Feynman rules:

2) For each vertex $\times$ you get $-\mathrm{i}\lambda \int \mathrm{d}^4 x$.

1) For each internal line from x to y you get $D_F(x-y)$.

3) For each external line to x you get $\exp(-\mathrm{i} p\cdot x)$.

4) Divide by symmetry factor.

Comments. After all external particle operators have eliminated factors of φ, there will remain

$$\langle 0|T\varphi(x_1)...\varphi(x_n)|0\rangle$$

This is evaluated by expressing each factor φ in annihilation and creation parts $\varphi(x)=\varphi_+(x)+\varphi_-(x)$ and moving φ_+ to the right, φ_- to the left, picking up commutators each time a φ_+ meets a φ_-.

Example. If $t_1>t_2>...>t_n$, then if $i<j$ there will occur terms with factor $[\varphi_+(x_i), \varphi_-(x_j)] = D_F(x_i-x_j)$ when $t_i>t_j$.

Wick's theorem says that $\langle 0|T\varphi(x_1)...\varphi(x_n)|0\rangle =$ sum of all possible contractions tere are.

Example: $\langle 0|T\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \langle\text{fig}\rangle$

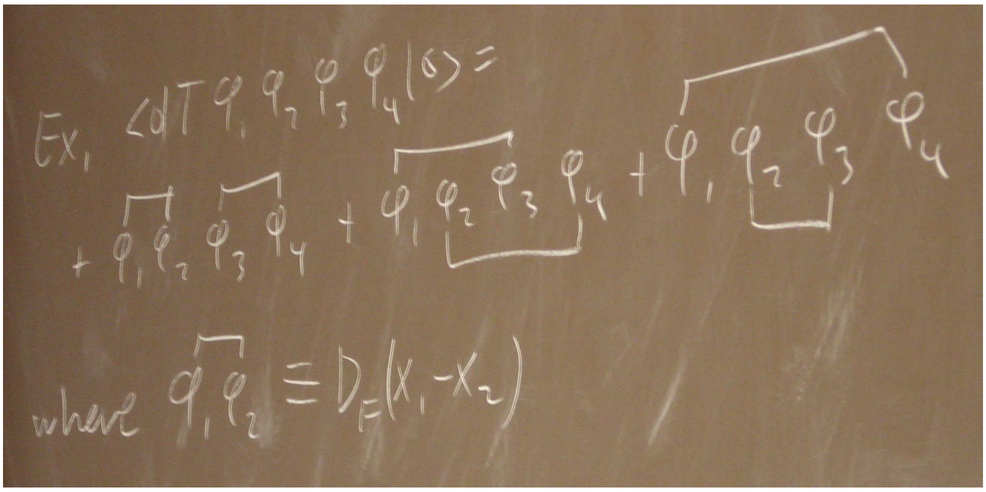

Figure 3. $\langle 0|T\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = ...$

where $\overline{\varphi_1\varphi_2} \equiv D_F(x_1-x_2)$.

$$\varphi(x)=\int \frac{\mathrm{d}^3 k}{(2\pi)^3\sqrt{2E_k}}\left(a(k)\,\mathrm{e}^{-\mathrm{i}k\cdot x}+a^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right)$$

$$\left[\varphi(x), a^\dagger(p)\right]=\int \frac{\mathrm{d}^3 k}{(2\pi)^3\sqrt{2E_k}}(2\pi)^3\,\delta^3(k-p)\,\mathrm{e}^{-\mathrm{i}k\cdot x}=\frac{1}{\sqrt{2E_p}}\,\mathrm{e}^{-\mathrm{i}p\cdot x}$$

Consider $p_1+p_2\rightarrow q_1+q_2$ to lowest order (λ^2) in $\frac{\lambda}{3!}\varphi^3$ theory.

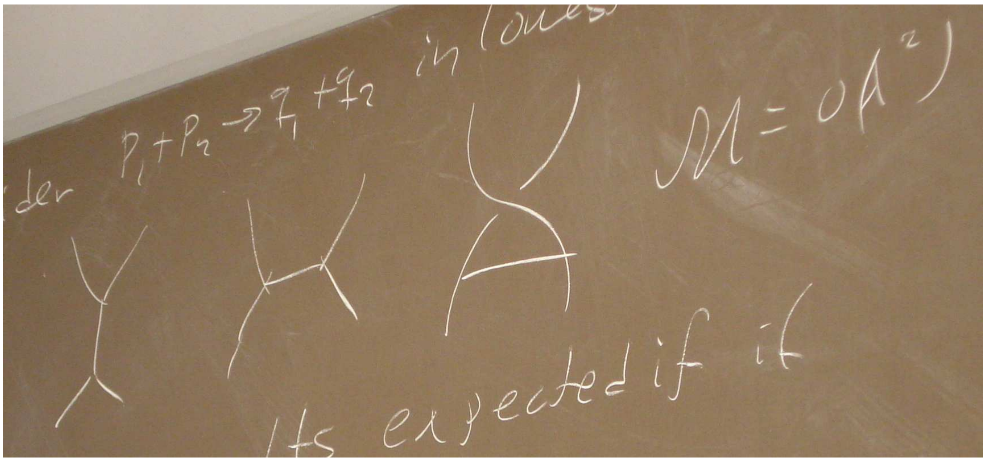

Figure 4. $\mathcal{M} = \mathcal{O}(\lambda^2)$.

Good idea: check that σ is as I claimed by going through the box quantisation scheme.

More accurate results are expected if higher oder graphs are added. They are obtained by adding more vertices and lines in the above graphs, e.g. to order λ^4:

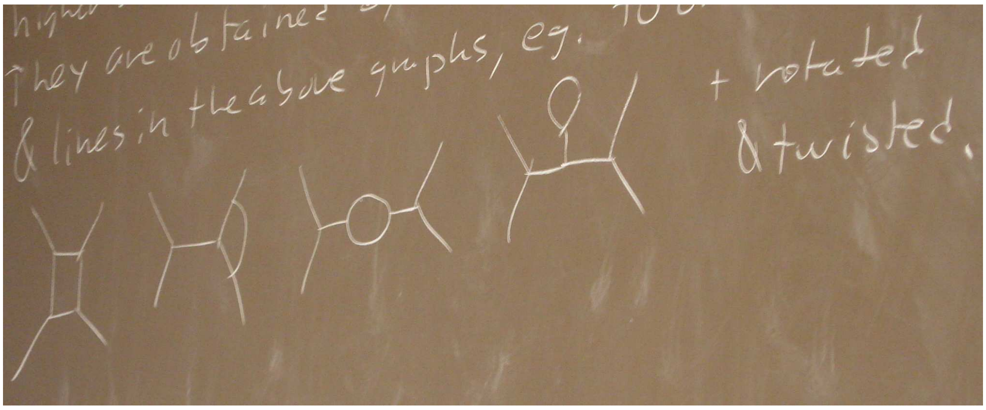

Figure 5. Graphs to order λ^4. Lots of possibilities. It is easy to forget something.

Disconnected graphs should not be added.

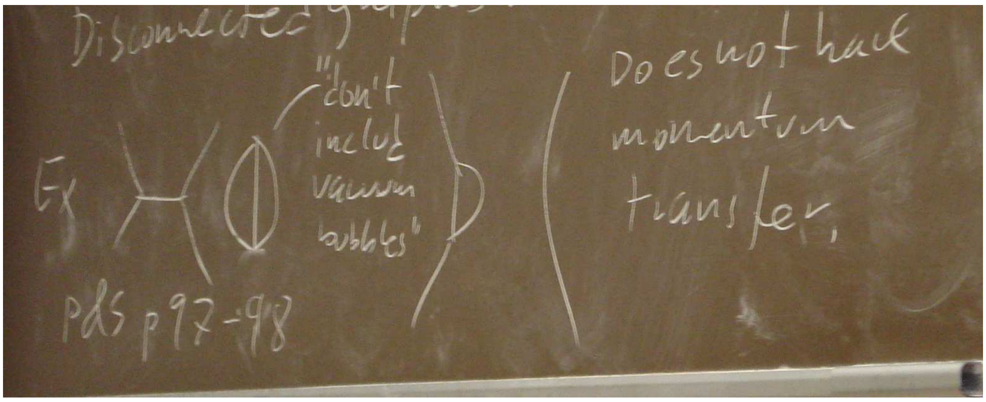

Figure 6. Disconnected graphs. Don't include vaccum bubbles, or graphs with no exchange of momentum.

Only *amputated* graphs contribute.

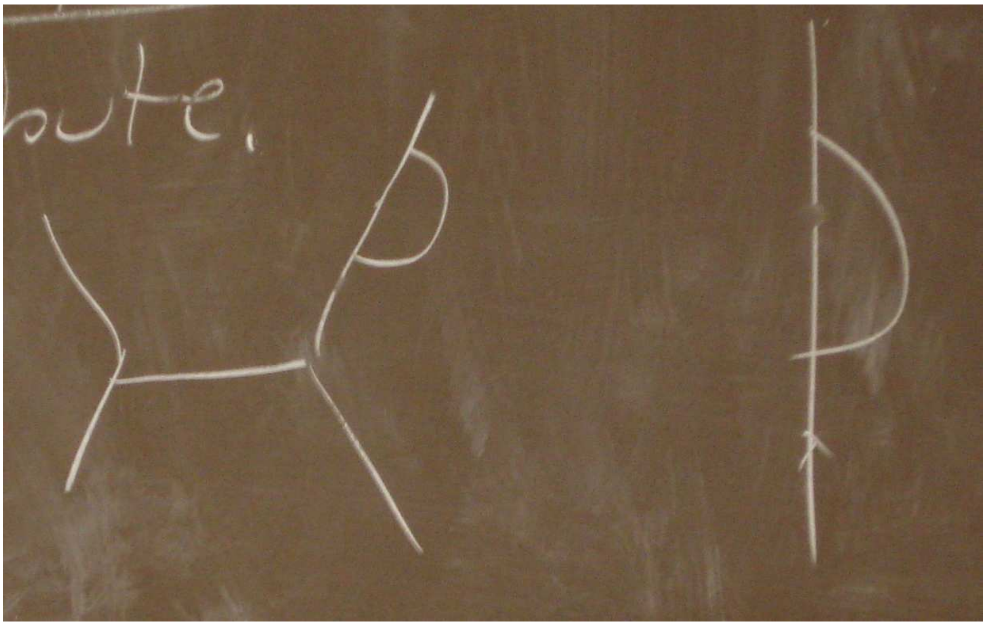

Figure 7. This should not be included. Only amputated graphs are to be included.

Point 6: symmetry factors. They are a bit difficult to explain. Perhaps you should tro to get them right in each case by yourself. Rule is, divide by the order of the symmetry of the diagram.

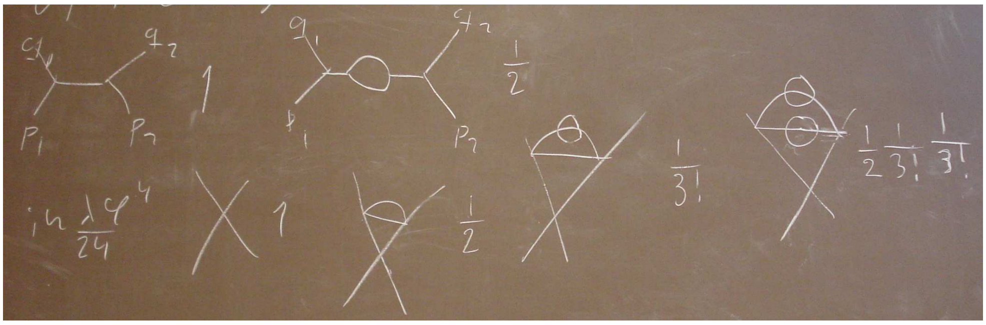

Figure 8. Symmetry factors. In the loop, the lower part and the upper part can be exchanged; two lines can be exchanged: symmetry factor 1/2.

Graph terminology: Graphs can be divided into complementary sets:

Connected + disconnected.

Tree graphs + loop graphs.

Other interacting quantum field theories

Example. Gauging free Dirac fermion theory produces gauge theory, interacting theory.

$$\mathcal{L}_0 = \bar{\psi}\,(\not{\partial} - m)\,\psi$$

It has a continuous global symmetry $\delta\psi = -\,\mathrm{i}\,\varepsilon\,\psi$, for constant parameter ε. The symmetry can be made local, $\varepsilon \to \varepsilon(x)$ by adding a gauge field $A_\mu(x)$ and replacing derivatives with covariant derivatives $\partial_\mu\psi \to \mathrm{D}_\mu\psi \equiv (\partial_\mu + \mathrm{i}\,A_\mu)\psi$.

$$\delta\partial_\mu\psi = -\,\mathrm{i}\,\varepsilon\,\partial_\mu\psi - \mathrm{i}\varepsilon_{,\mu}\psi$$

$$\delta\mathrm{D}_\mu\psi = -\,\mathrm{i}\varepsilon\mathrm{D}_\mu\psi$$

$$\delta A_\mu = \partial_\mu\varepsilon$$

Gauge invariant, Lagrangian density:

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\not{\mathrm{D}} - m)\,\psi - \frac{1}{4\,e^2}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \delta F_{\mu\nu} = 0.$$

Usually A_μ is redefined, $A_\mu \to e\,A_\mu \Rightarrow$

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\not{\partial} - m)\,\psi - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

This is QED: quantum electrodynamics, for electrons, positrons and photons.

$$\psi(x) = (2\pi)^{-3}\int \frac{\mathrm{d}^3p}{\sqrt{2\,E_p}}\sum_s \left(a_s(\boldsymbol{p})u_s(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + b_s^\dagger(p)v_s(p)\,\mathrm{e}^{\mathrm{i}p\cdot x}\right)$$

$$A^\mu(x) = (2\pi)^{-3}\int \frac{\mathrm{d}^3p}{\sqrt{2E_p}}\sum_s \left(a_s(\boldsymbol{p})\varepsilon_s^\mu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}p\cdot x} + a_s^\dagger(\boldsymbol{p})\varepsilon_s^{*\mu}(\boldsymbol{p})\,\mathrm{e}^{\mathrm{i}p\cdot x}\right)$$

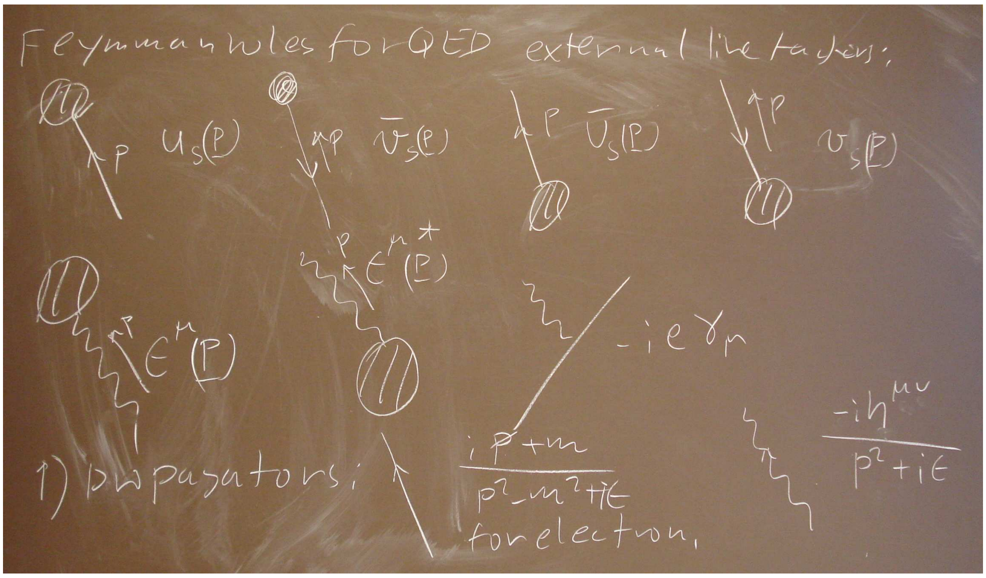

Figure 9. Feynman rules for QED.

Feynman rules for QED:

1.a Propagator for electron and positron:

$$\frac{\mathrm{i}\not{p} + m}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

1.b Propagator for photon:

$$\frac{-\,\mathrm{i}\eta^{\mu\nu}}{p^2+\mathrm{i}\varepsilon}$$

2. Vertex.
3. External line factors.
4. Momentum conservation at each vertex.
5. Integrate over undetermined momenta.
6. Figure out the overall sign.