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Today: Interacting quantum field theories. The past two weeks have been about free quantum theories: the scalar field and the spin $\frac{1}{2}$ fields. Now it is time for interacting fields: chapter 4 in Peskin&Schroeder. Our treatment will differ a bit from that in the book. There is much more formalism than we will treat here.

EXAMPLE: Scalar φ^3 theory. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\big(\partial^\mu\varphi\,\partial_\mu\varphi - m^2\varphi^3\big) - \frac{1}{6}\lambda\,\varphi^3$$

λ is the coupling constant, assumed small (so that one can do perturbation theory). The factor $1/6 = 1/3!$ is conventional. (The number 3 in $1/3!$ comes from the exponent in φ^3.) The Hamiltonian is:

$$H = \int \mathrm{d}^3x \left[\frac{1}{2}\Big(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\Big) + \frac{1}{6}\lambda\varphi^3\right] = H_0 + H_I$$

The Hamiltonian gives us the time evolution operator $\exp(-\mathrm{i}\, H\, t)$.

Let us express this in terms of creation and annihilation operators:

$$H_0 = \sum_p E_p\, a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}), \quad H_I \sim \left(a + a^\dagger\right)^3 \text{ (very schematically)}$$

In H_I above we are just keeping track of the number of creation and annihilation operators.

A two-particle state can be described by $|p_1 p_2\rangle = a^\dagger(\boldsymbol{p}_1)\ a^\dagger(\boldsymbol{p}_2)|0\rangle$. Time development with H_0 just gives it a phase:

$$\exp(-\mathrm{i}\, H_0 t)|p_1 p_2\rangle = \exp(-\mathrm{i}\,(E_1 + E_2)t)|p_1 p_2\rangle$$

Assuming λ to be small, we can write $\exp(-\mathrm{i}\, H_I\, t) \approx 1 + \mathrm{i}\, t\, H_I$. (We can't really factor $\exp(-\mathrm{i} H\, t)$ into a free-particle part and an interaction part, because things don't commute. But never mind that now.) H_I can do interesting things. It will contain terms like:

$$-\mathrm{i}\, t\, a^\dagger(p_3) a^\dagger(p_4) a(p_5)\, a^\dagger(p_1)\, a^\dagger(p_2)|0> =$$

Take annihilation operators to the right and the creation operators to the left.

$$= -\mathrm{i}\, t\, a^\dagger(p_3)\, a^\dagger(p_4)\, (\delta(p_5 - p_1) a^\dagger(p_2) + a^\dagger(p_1)\, \delta(p_5 - p_2))|0\rangle$$

$$\int \mathrm{d}^3 x \quad \Rightarrow \quad \text{momentum conservation, } p_3 + p_4 = p_5$$

Feynman graph picture

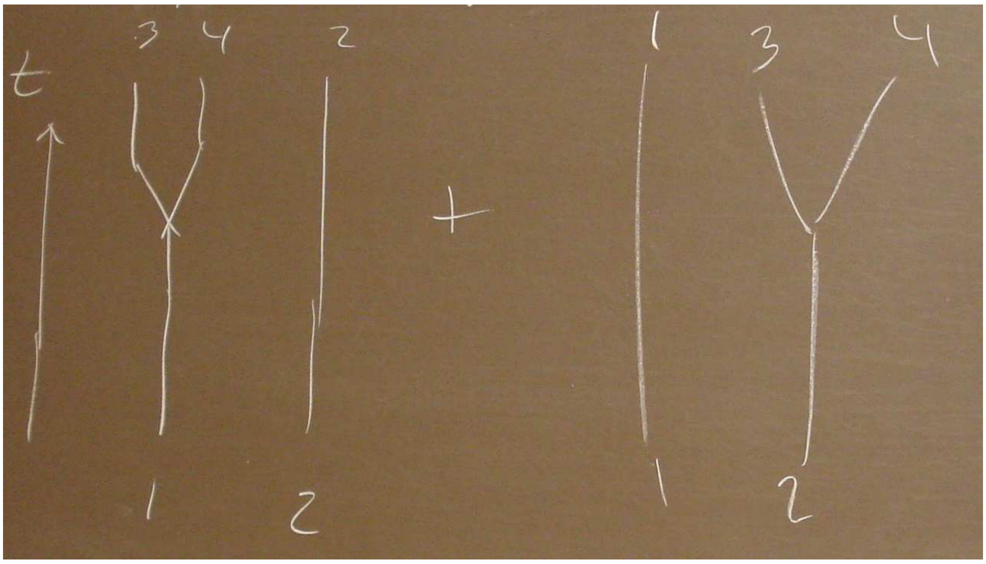

Figure 1. Feynman diagram. Each particle is a line. Each interaction is a vertex. In φ^3 theory always three lines meet at vertices.

This violates energy conservation, since all particles have the same mass. But going to higher order, $\exp(-\mathrm{i}\,H_I t) \approx 1 + \mathrm{i}\,t\,H_I + \frac{1}{2}(-\mathrm{i}\,t\,H_I)^2$, we get another vertex. Time development during long time will produce the requirement of energy conservation, and forbid processes where one particle produces two, kinematically. But to next order in perturbation theory one can have:

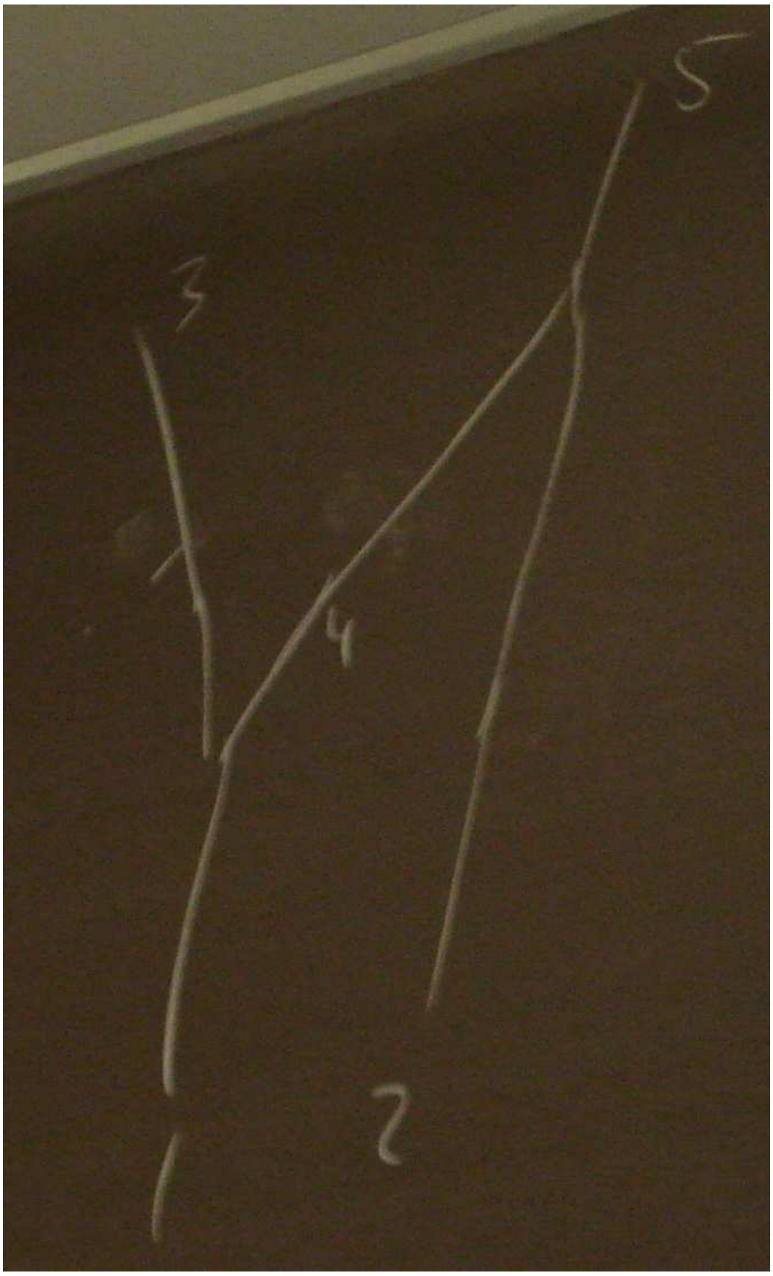

Figure 2. Two vertices.

$$-\mathrm{i}\,t\,a_5^\dagger\,a_6\,a_7\Big(-\mathrm{i}\,t\,a_3^\dagger a_4^\dagger a_2^\dagger\Big)|0\rangle$$

Note that there is momentum conservation at each vertex. But particle four transfers momentum from particle one to particle two $\Rightarrow$ particles three and five have different momenta than particles one and two.

This Feynman graph describes *elastic scattering*: $1 + 2 \rightarrow 3 + 5$. Elastic means that the same particles come out as come in. There are three graphs contributing to this process in lowest order.

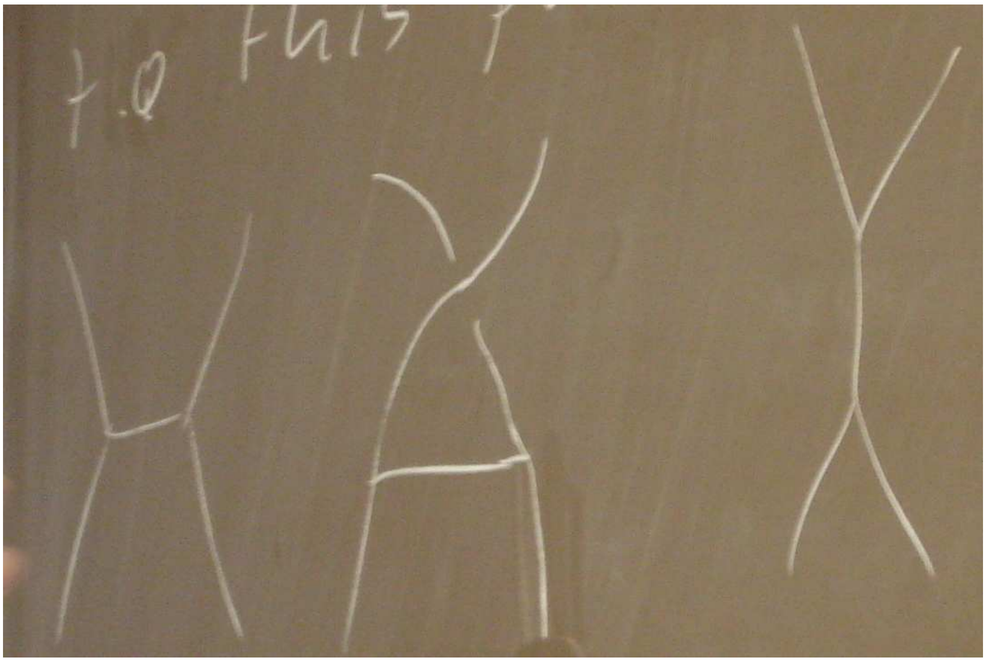

Figure 3. Three graphs: check this.

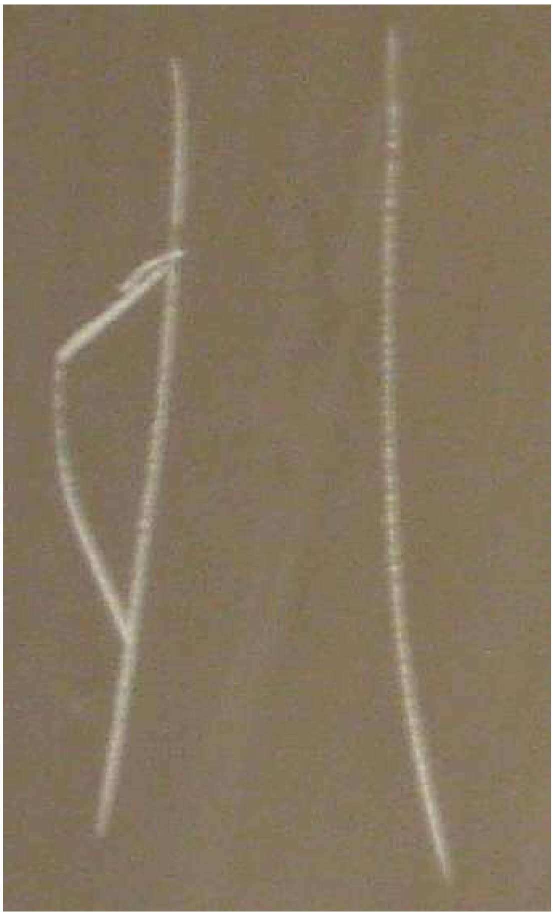

Figure 4. Note: This does not produce any scattering. There is no momentum transfer.

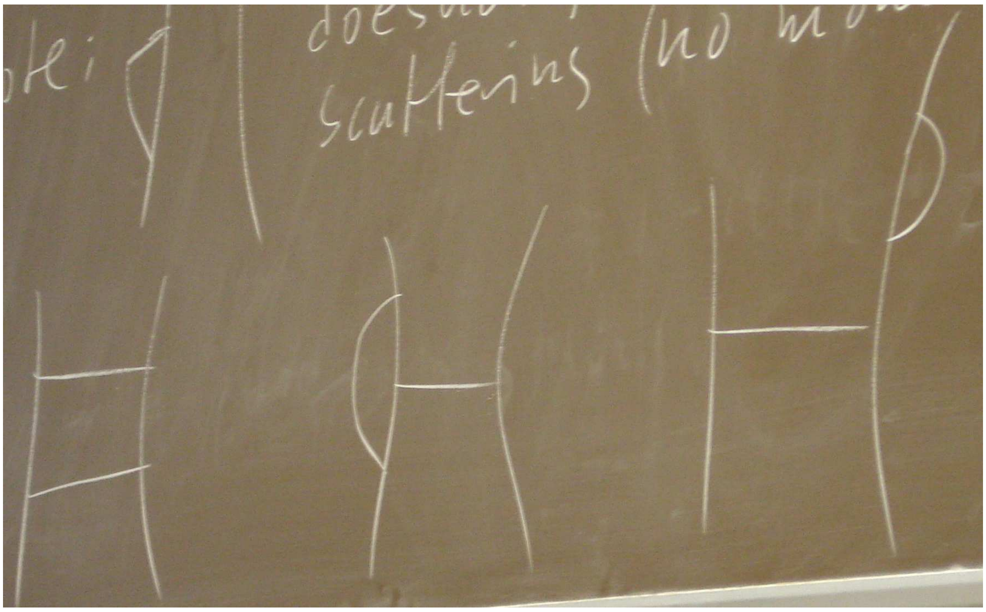

Figure 5. These are of higher order, but should in principle be included.

Free $\mathcal{L}, H_0$ give particle lines. H_I gives vertices. φ^n gives n particle vertices.

Computing a cross section

Example: $p_1 + p_2 \to q_1 + \cdots + q_n$.

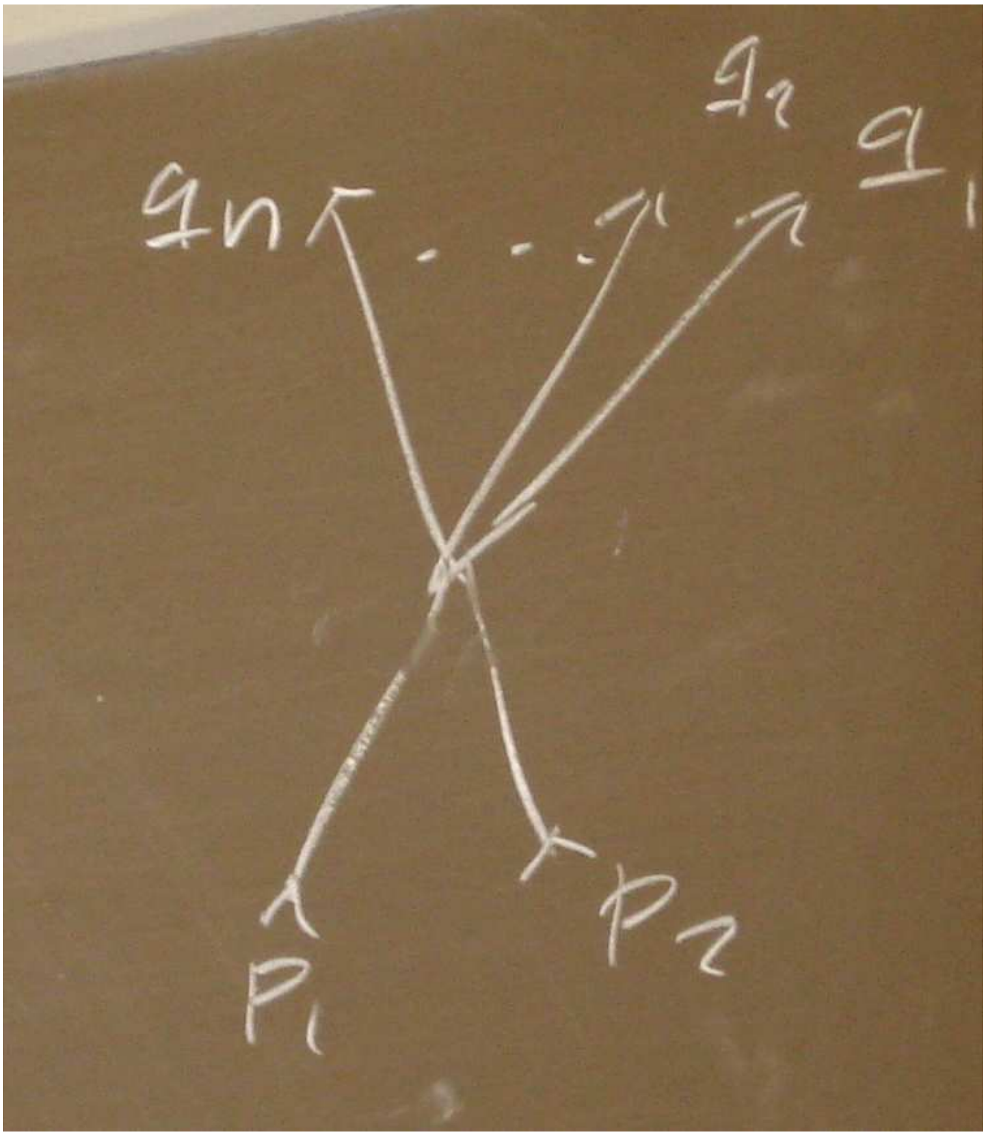

Figure 6. $p_1 + p_2 \to q_1 + \cdots + q_n$

$$\mathcal{L} = \frac{1}{2}\Big((\partial\varphi)^2 - m^2\varphi^2\Big) - \frac{\lambda}{(n+2)!}\varphi^{n+2}$$

Imagine space=box, side L, volume $V = L^3$, existing during time 0 to T.

$$\varphi(x) = V^{-1/2}\sum_p \frac{1}{\sqrt{2E_p}}\left(a(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger(\boldsymbol{p})\,\mathrm{e}^{\mathrm{i}p\cdot x}\right)$$

$$\boldsymbol{p} = \frac{2\pi}{L}(n_1, n_2, n_3), \quad p^0 = \sqrt{m^2 + \boldsymbol{p}^2}, \quad \left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p})\right] = \delta_{\boldsymbol{p}\boldsymbol{p}'}$$

At time 0 our state is $|p_1, p_2\rangle = a^\dagger(\boldsymbol{p})\, a^\dagger(\boldsymbol{p})|0\rangle$.

$$\langle q_1, \ldots, q_n|\exp(-\mathrm{i}\, H T)|p_1, p_2\rangle$$

Probability that the state time T is $|q_1, \ldots, q_n\rangle$ is $|S_{qp}|^2$ where

$$S_{qp} = \langle q_1, \ldots, q_n|\exp(\mathrm{i} H_0 T) \exp(-\mathrm{i}\, H T)|p_1 p_2\rangle$$

$$U(T, 0) = \exp(\mathrm{i} H_0 T) \exp(-\mathrm{i}\, H T)$$

$U(T, 0)$ is called interaction picture time evolution operator. Property:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(t, 0) = \exp(\mathrm{i}\, H_0 t) \underbrace{(H - H_0)}_{H_I} \exp(-\mathrm{i} H t) = \underbrace{\exp(\mathrm{i} H_0 t)\, H_I \exp(-\mathrm{i} H_0\, t)}_{H_{II}} \underbrace{\exp(\mathrm{i}\, H_0 t) \exp(-\mathrm{i} H\, t)}_{= U(t, 0)}$$

$$H_{II}(t) = \frac{\lambda}{(n+2)!}\, \varphi^{n+2}(x, t)_{\text{free}}$$

Solution of differential equation:

$$U(t, 0) = T \exp\left(-\mathrm{i} \int_0^t H_{II}(t')\mathrm{d}t'\right) \quad \text{(time-ordered product)}$$

$$S_{qp} = \langle q_1, \ldots, q_n| -\mathrm{i} \frac{\lambda}{(2+n)!} \int_{t=0}^{t=T} \varphi_{\text{free}}^{2+n}\, \mathrm{d}^4 x\, |p_1, p_2\rangle =$$

$$= \langle 0|a(q_1) \ldots a(q_n) \quad \ldots \quad a^\dagger(p_1) a^\dagger(p_2)|0\rangle =$$

$$= -\mathrm{i}\lambda \int \mathrm{d}^4 x\, V^{-\frac{n+2}{2}} \left(\prod_{i=1}^n \frac{1}{\sqrt{2E_i}} \exp(\mathrm{i}\, q \cdot x)\right) \frac{1}{\sqrt{2E_1 \cdot 2E_2}} \exp(-\mathrm{i}(p_1 + p_2) \cdot x) =$$

$$= -\mathrm{i}\lambda^{-n/2} \int_0^T \mathrm{d}t \left(\prod_i \frac{1}{\sqrt{2E_i}}\right) \frac{1}{\sqrt{2E_1 \cdot 2E_2}}\, \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2} - \sum_i \boldsymbol{q}_i \exp\left(\sum_i E_i - E_{p_1} - E_{p_2}\right)$$

Probability

$$|S_{qp}|^2 = \lambda^2\, V^{-n} \frac{\delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2 - \sum_i \boldsymbol{q}_i}}{2\, E_{p_1} \cdot 2 E_{p_2} \cdot \left(\prod_i 2\, E_i\right)} \left| \int_0^T \mathrm{d}t \exp\left(-\mathrm{i}\left(E_{p_1} + E_{p_2} - \sum_i E_i\right) t\right) \right|^2$$

We are interested in not just one final state, but all in a volume element of momentum space $\Rightarrow$ multiply by a factor $V/(2\pi)^3 \cdot \mathrm{d}^3 q_i$ for each final particle, except the last one. (The last one is fixed by momentum conservation, the δ-symbol.)

$$\Rightarrow \text{factor} = \prod_{i=1}^{n-1} \frac{V}{(2\pi)^3} \mathrm{d}^3 q_i = \frac{V^{-1}}{\left((2\pi)^3\right)^{n-1}} \left(\prod_{i=1}^n \mathrm{d}^3 q_i\right) \delta^3\left(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \sum_i \boldsymbol{q}_i\right)$$

We want the differential cross section $\mathrm{d}\sigma$. Remember the definition of cross section:

$$\text{number of scattered particles} = \sigma \cdot |\Delta \boldsymbol{v}| \cdot \text{time} \cdot \text{beam density}$$

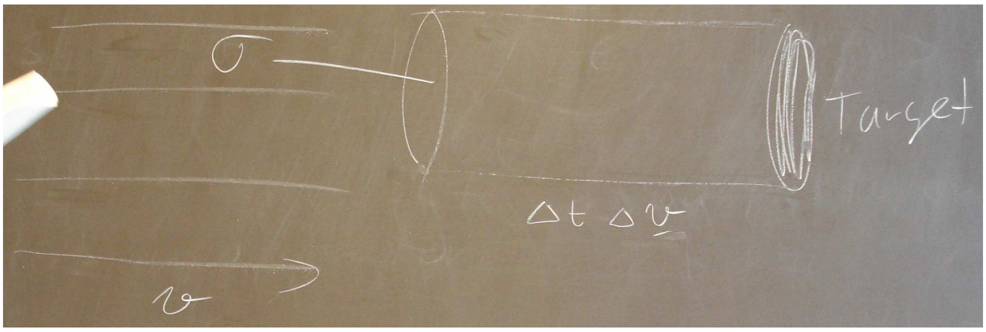

Figure 7. Cross section.

To get $\mathrm{d}\sigma$ we therefore divide our probability by

$$|\Delta v| = \left|\frac{\boldsymbol{p}_1}{E_1} - \frac{\boldsymbol{p}_2}{E_2}\right|, \quad \Delta t = T, \quad \text{beam density} = \frac{1}{V}$$

Time integral:

$$\left|\int_{-T/2}^{T/2} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}Et}\right|^2 = \left|\frac{2}{E}\sin\left(\frac{ET}{2}\right)\right|^2 = \frac{4}{E^2}\sin^2\left(\frac{ET}{2}\right) \approx C\,\delta(E)$$

$$C = \int_{-\infty}^{\infty} \mathrm{d}E\, \frac{4}{E^2}\sin^2\left(\frac{ET}{2}\right) = T\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{4}{x^2}\sin^2\left(\frac{x}{2}\right) = \left[\begin{array}{c}\text{by complex contour}\\ \text{integration}\end{array}\right] = T\cdot 2\pi$$

Putting everything together:

$$\mathrm{d}\sigma = \lambda^2 \frac{1}{|\Delta\boldsymbol{v}|} \frac{1}{2\,E_{p_1}\cdot 2\,E_{p_2}} \prod_{i=1}^{n} \frac{\mathrm{d}^3 q_i}{(2\pi)^3\cdot 2E_{q_i}} (2\pi)^4\, \delta^4\left(p_1 + p_2 - \sum_i q_i\right)$$

This is equation 4.79 in Peskin&Schroeder, with $\mathcal{M} = -\lambda$.

Example. Take $n = 2, \boldsymbol{p}_2 = -\boldsymbol{p}_1 = \boldsymbol{p}$.

$$\mathrm{d}\sigma = \lambda^2 \frac{1}{2|\boldsymbol{p}|/E} \cdot \frac{1}{2\,E\cdot 2\,E} \cdot \frac{\mathrm{d}^3 q}{(2\pi)^3 (2E)^2} \cdot 2\pi\,\delta(2E_p - 2E_q)$$

$$\int \mathrm{d}^3 q = \frac{1}{2}\cdot 4\pi \int q^2 \mathrm{d}q = 2\pi \int q\,E_q\,\mathrm{d}E_q$$

$$\sigma = \lambda^2 \cdot \frac{1}{2|\boldsymbol{p}|\,4E} \frac{\frac{1}{2}\,4\,\pi\,q\,E_q}{(2\pi)^3 (2E)^2}\, 2\pi\, \frac{1}{2} = \frac{\lambda^2}{32\pi (2E)^2}$$