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Last time was the quantisation of the Dirac field, and some of its most basic properties. You noticed that it becomes a bit messy with long expression. There is a lot more to be said, and chapter 3 in Peskin&Schroeder is a heavy chapter. We won't go into all of that, but a few things remain to be said.

Symmetries and charges

General scheme (schematically): Assume there is a symmetry under $\varphi \to \varphi + \delta\varphi$:

$$\delta\mathcal{L} = \mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = \partial_\lambda J^\lambda$$

If the variation of the Lagrangian is a divergence, it only changes the action by boundary terms and the equations of motion are not affected, as expected from a symmetry.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\,\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi_{,\lambda}}\,\delta\varphi_{,\lambda} = \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\lambda\frac{\partial\mathcal{L}}{\partial_{,\lambda}}\right)}_{=0,\,\mathrm{eq.\,of\,motion}}\delta\varphi + \partial_\lambda\!\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\lambda}}\,\delta\varphi\right)$$

$\Rightarrow$ current conservation: $\partial_\lambda j^\lambda = 0$

$$j^\lambda = \frac{\partial\mathcal{L}}{\partial\varphi_{,\lambda}}\,\delta\varphi - J^\lambda$$

$$Q = \int \mathrm{d}^3x\, j^0$$

generates symmetry $[\mathrm{i}Q, \varphi] = \delta\varphi$.

Example 1: Charge in Dirac theory:

$$\psi \to \mathrm{e}^{-\mathrm{i}\varepsilon}\psi, \quad \delta\psi = -\mathrm{i}\varepsilon\,\psi, \quad J^\lambda = 0, \quad j^\lambda = \mathrm{i}\,\bar{\psi}\,\gamma^\lambda(-\mathrm{i}\psi) = \bar{\psi}\,\gamma^\lambda\psi$$

since

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\,\gamma^\lambda\partial_\lambda - m\right)\psi, \quad \frac{\partial\mathcal{L}}{\partial\psi_{,\lambda}} = \mathrm{i}\bar{\psi}\gamma^\lambda.$$

We get the charge

$$Q = \int \mathrm{d}^3x\, \psi^\dagger\psi.$$

Write in terms of creation and annihilation operators:

$$\psi(x) = (2\pi)^{-3}\int \frac{\mathrm{d}^3p}{\sqrt{2E_p}}\sum_s \left(a_s(p)\,u_s(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x} + b_s^\dagger(p)\,v_s(p)\,\mathrm{e}^{\mathrm{i}p\cdot x}\right)$$

Use relation 1 from before: $u_s^\dagger(p)u_{s'}(p) = 2E_p\delta_{s s'}$.

$$Q = (2\pi)^{-3}\int \mathrm{d}^3p\sum_s \left(a_s^\dagger(\boldsymbol{p})\,a_s(\boldsymbol{p}) + b_s(\boldsymbol{p})\,b_s^\dagger(\boldsymbol{p})\right) =$$

$$= (2\pi)^{-3}\int \mathrm{d}^3p\sum_s \left(a_s^\dagger(\boldsymbol{p})\,a_s(\boldsymbol{p}) - b_s^\dagger(\boldsymbol{p})\,b_s(\boldsymbol{p}) + (2\pi)^3\,\delta^3(\boldsymbol{0})\right)$$

The last term is a vacuum charge. An infinite vacuum charge. We don't want it, so we throw it away:

$$Q = (2\pi)^{-3} \int \mathrm{d}^3 p \sum_s \left(a_s^\dagger(\boldsymbol{p})\, a_s(\boldsymbol{p}) - b_s^\dagger(\boldsymbol{p})\, b_s(\boldsymbol{p}) \right)$$

We can replace the charge $Q = \int \mathrm{d}^3 x\, \psi^\dagger \psi$ by

$$Q = \int \mathrm{d}^3 x\, \frac{1}{2}\left(\psi^\dagger \psi - \psi\, \psi^\dagger \right)$$

There is no reason why $\psi^\dagger$ should stand to the left, they are equal fields. This way, one takes away the vacuum charge.

Example 2: Translation symmetry. $\delta\varphi = \varepsilon^\rho \partial_\rho \varphi$, which corresponds to $x^\rho \to x^\rho \pm \varepsilon^\rho$ (never mind the sign right now).

$$\varepsilon_\rho J^{\lambda\rho} = \varepsilon_\rho\, \eta^{\lambda\rho} \mathcal{L}, \quad \varepsilon_\rho \partial_\lambda J^{\rho\lambda} = \varepsilon_\rho\, \partial^\rho \mathcal{L}$$

$\mathcal{L} = 0$ by equations of motion in Dirac theory.

Conserved current $T^{\lambda\rho} = \mathrm{i}\, \bar{\psi} \gamma^\lambda \psi'^\rho$. Energy-momentum density. Energy-momentum vector:

$$p^\rho = \int \mathrm{d}^3 x\, \psi^\dagger \mathrm{i}\, \partial^\rho \psi$$

Example 3: Lorentz transformations.

$$\delta\psi = -\frac{\mathrm{i}}{2}\, \omega_{\mu\nu} (J^{\mu\nu} + S^{\mu\nu}) \psi$$

$$M^{\lambda\mu\nu} = \mathrm{i} \bar{\psi} \gamma^\lambda (-\mathrm{i}\, J^{\mu\nu} - \mathrm{i}\, S^{\mu\nu}) \psi = \mathrm{i} \bar{\psi} \left(x^\mu \partial^\nu - x^\nu \partial^\mu + \frac{1}{4} [\gamma^\mu, \gamma^\nu] \right) \psi =$$

$$= \underbrace{x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}}_{\text{space part}} + \underbrace{\bar{\psi} \gamma^\lambda \frac{\mathrm{i}}{4} [\gamma^\mu, \gamma^\nu] \psi}_{\text{spin part}}$$

When you integrate it you get conserved charge:

$$M^{\mu\nu} = \int \mathrm{d}^3 x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu} + \psi^\dagger \frac{\mathrm{i}}{4} [\gamma^\mu, \gamma^\nu] \psi \right)$$

Operator generator of Lorentz transformation. This is Hermitian, so that Lorentz transformations acting on the Hilbert space are unitary operators. See Peskin&Schroeder page 59 about this.

Remark:

$$S^{12} = \frac{\mathrm{i}}{4} [\gamma^1, \gamma^2] = \frac{\mathrm{i}}{4} \begin{pmatrix} \sigma^1 \bar{\sigma}^2 - \sigma^2 \bar{\sigma}^1 & \\ & \bar{\sigma}^1 \sigma^2 - \bar{\sigma}^2 \sigma^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix}$$

Hermitian. $\Rightarrow M^{12}_{\text{spin}}$ is Hermitian.

$$S^{03} = \frac{\mathrm{i}}{4} [\gamma^0, \gamma^3] = \frac{\mathrm{i}}{2} \begin{pmatrix} -\sigma^3 & \\ & \sigma^3 \end{pmatrix}$$

This is non-Hermitian. $\Rightarrow M^{03}_{\text{spin}}$ is non-Hermitian. (The spinor representation is not a unitary representation.) We have to check the space part too, and it turns out that that is not Hermitian either.

$$M^{03}_{\text{space}} - \big(M^{03}_{\text{spa}}\big)^{\dagger} = \int \mathrm{d}^3x\, \big(-x^3\bar{\psi}\,(-\mathrm{i}\gamma^k\partial_\lambda + m)\psi - \text{Hermitian conjugate}\big) =$$

$$= \mathrm{i}\int \mathrm{d}^3x\, x^3\partial_k\bar{\psi}\gamma^k\psi = -\mathrm{i}\int \mathrm{d}^3x\, \bar{\psi}\gamma^3\psi$$

The non-Hermicity of the space part exactly takes out that of the spin part.

"Then I can make you even more confused, I think." — non-symmetric energy-momentum tensor in field theory. $T^{\lambda\rho}_{\text{Noether}} = \mathrm{i}\bar{\psi}\gamma^\lambda\partial^\rho\psi$ is not symmetric, $T^{\lambda\rho} \neq T^{\rho\lambda}$, but $T^{\lambda\rho}_{\text{gravity}}$ must be symmetric. Srednicki page 230 (linked from course home page) describes how $T^{\lambda\rho}_{\text{Noether}}$ can be modified by adding terms into $\theta^{\lambda\rho}$ (symmetric). Belinfante tensor. Srednicki claims it equals $T^{\lambda\rho}_{\text{gravity}}$.

$T^{\lambda\rho}$ and $\theta^{\lambda\rho}$ produce the same p^ρ. Moreover

$$\Xi^{\lambda\rho\nu} = x^\mu\theta^{\lambda\nu} - x^\nu\theta^{\lambda\mu}$$

No spin term. Ξ produces the same Lorentz generators $M^{\mu\nu}$ as $M^{\lambda\mu\nu}$ does.

$$M^{12}_{\text{spin}} = \int \mathrm{d}^3x\psi^\dagger S^{12}\psi = \int \mathrm{d}^3x\, \psi^\dagger\frac{1}{2}\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\psi$$

$$M^{12}_{\text{spin}}|\boldsymbol{p}s\rangle = M^{12}_{\text{spin}}\, a^\dagger_s(\boldsymbol{p})|0\rangle = \Big[M^{12}_{\text{spin}}, a^\dagger_s(\boldsymbol{p})\Big]|0\rangle = \int \mathrm{d}^3x\, \psi^\dagger(x)\,\frac{1}{2}\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\Big[\psi, a^\dagger_s(\boldsymbol{p})\Big]_+ = \ldots$$

$$= \frac{1}{2E_p}\sum_{s'} a^\dagger_{s'}u^\dagger_{s'}(p)\, S^{12}u_s(\boldsymbol{p})|0\rangle = \pm\frac{1}{2}a^\dagger_s|0\rangle$$

if $\boldsymbol{p} = 0$ or if $\boldsymbol{p} = |\boldsymbol{p}|\,\hat{z}$.

Conclusion, at momentum $\boldsymbol{p} = |\boldsymbol{p}|\,\hat{z}$ create a particle of spin s_z:

$$\begin{array}{cc} a^\dagger_1 & \frac{1}{2} \\ a^\dagger_2 & -\frac{1}{2} \\ b^\dagger_1 & -\frac{1}{2} \\ b^\dagger_2 & \frac{1}{2} \end{array}$$

$$u_s(\boldsymbol{p} = 0) = \sqrt{2}m\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Charge conjugation:

Remember:

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Define charge conjugated spinor

$$\psi^c = -\,\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} -\,\mathrm{i}\,\sigma^2\psi_R^* \\ \mathrm{i}\sigma^2\psi_L^* \end{pmatrix}$$

ψ^c transforms like ψ under Lorentz transformations, i.e. $-\,\mathrm{i}\sigma^2\psi_L^*$ transforms like ψ_R etc.

Basic reason: $\left(\sigma^1,\sigma^2,\sigma^3\right)^* = \left(\sigma^1,-\,\sigma^2,\sigma^3\right)$.

$$\sigma^2\,\sigma^{\mu *}\left(\sigma^2\right)^{-1} = \bar{\sigma}^{\,\mu}, \quad \gamma^2[\gamma^\mu,\gamma^\nu]^*\left(\gamma^2\right)^{-1} = [\gamma^\mu,\gamma^\nu]$$

$\psi \to \psi^c$ is a symmetry of Dirac theory. Charge conjugation $a_1^\dagger(\boldsymbol{p}) \leftrightarrow b_2^\dagger(\boldsymbol{p})$.

Dirac fermion bilinears $\bar{\psi}_a\psi_b, a, b = 1, \ldots, 4$. Totally 16. Normally written in terms of $\bar{\psi}\psi$, $\bar{\psi}\gamma^m\psi$, $\bar{\psi}\gamma^\mu\gamma^\nu\psi$: transform under Lorentz transformations as scalar, vector, tensor, etc.

It is enough to take totally antisymmetric tensor, because $\bar{\psi}\gamma^\mu\gamma^\nu + \bar{\psi}\gamma^\nu\gamma^\mu\psi = \bar{\psi}\psi 2\eta^{\mu\nu}$. $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\,[\gamma^\mu,\gamma^\nu]\,\psi$.

Define a matrix called γ^5:

$$\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\,\gamma^2\,\gamma^3$$

$\bar{\psi}\psi$		$\bar{\psi}\gamma^\mu\psi$		$\bar{\psi}\,[\gamma^\mu,\gamma^\nu]\,\psi$		$\bar{\psi}\gamma^\mu\gamma^5\psi$		$\bar{\psi}\gamma^5\psi$		
scalar		vector		tensor		pseudovector		pseudoscalar		
1	$+$	4	$+$	$\frac{4\cdot 3}{2}$	$+$	4	$+$	1	$=$	16

Chiral projection operator:

$$\frac{1}{2}\left(1-\gamma^5\right)\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad \frac{1}{2}\left(1+\gamma^5\right)\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

Discrete symmetries in Quantum Field Theory: P, T, C.

The Lorentz group as a manifold has 4 components.

Specific Lorentz transformations you don't reach by exponentiating infinitesimal transformations:

$$\begin{array}{ll} P & (t,\boldsymbol{x}) \to (t,-\,\boldsymbol{x}) \\ T & (t,\boldsymbol{x}) \to (-\,t,\boldsymbol{x}) \\ PT & \end{array}$$

C is charge conjugation.

Discrete symmetries P, C, T can be all be broken in quantum field theory. But the combination PCT cannot be broken, which may be used to prove that the mass of the electron equals that of the positron.

$$\bar{\psi}\gamma^5\psi = \psi_L^\dagger\bar{\sigma}^{\,\mu}\,\psi_L + \psi_R^\dagger\sigma^\mu\psi_R$$

There is quantum field theory with only ψ_L (no ψ_R). It can have a kinetic term $\mathrm{i}\psi_L^\dagger\bar{\sigma}^{\,\mu}\partial_\mu\psi_L$. It can couple to a vector field. It cannot have Dirac mass term, since

$$m\bar{\psi}\psi = m\left(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R\right)$$

but it can have Majorana mass

$$m\bar{\psi}\psi, \quad \psi = \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix} = \text{Majorana spinor}$$

but then it cannot couple to a vector field.