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Today's subject is, mostly, the quantisation of the Dirac field $\psi_a(x)$. Last time we talked about the Lorentz group, the group of Lorentz transformations, in terms of infinitesimal transformations. All fields of nonzero spin are nontrivial representations of Lorentz transformations.

Lorentz transformation of ψ infinitesimally

$$\delta\psi = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}(S^{\mu\nu} + J^{\mu\nu})\psi$$

can be exponentiated:

$$\psi(x) \to \psi'(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x)$$

$\Lambda_{1/2}$ is another type of representation of the same transformation.

Last time:

$$\mathcal{L}_0 = \bar{\psi}\left(\mathrm{i}\not{\partial} - m\right)\psi$$

Euler-Lagrange equations:

$$\left(\mathrm{i}\not{\partial} - m\right)\psi = 0 \quad \Rightarrow \quad \left(\partial^2 + m^2\right)\psi_a = 0$$

Each component of ψ satisfies the Klein-Gordon equation.

We develop the Hamiltonian formalism for this Lagrangian, and replace the Poisson brackets by quantum mechanical commutators to quantise the field.

To solve this equation we make the ansatz $\psi(x) = u(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x}$. $p^2 - m^2 = 0 \Rightarrow p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$.

Ansätze:

$$\begin{array}{c} \psi(x) = u(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} \\ \psi(x) = v(\boldsymbol{p})\,\mathrm{e}^{\mathrm{i}p\cdot x} \end{array} \quad \Rightarrow \quad \begin{array}{c} (\not{p} - m)u(\boldsymbol{p}) = 0 \\ (\not{p} + m)v(\boldsymbol{p}) = 0 \end{array}$$

How to find u and v?

For $\boldsymbol{p} = \boldsymbol{0}$:

$$\not{p} - m = \begin{pmatrix} -m & 0 & m & 0 \\ 0 & -m & 0 & m \\ m & 0 & -m & 0 \\ 0 & m & 0 & -m \end{pmatrix}$$

$$u(\boldsymbol{0}) = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$

$$v(\boldsymbol{0}) = \sqrt{m}\begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta_1, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta_2$$

The $\boldsymbol{p} \neq \boldsymbol{0}$ solutions are obtained by boosting $\boldsymbol{p} = \boldsymbol{0}$ solutions.

Result:

$$u_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\;\xi_s \end{pmatrix}, \quad v_s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\begin{pmatrix} \eta_s \\ \eta_s \end{pmatrix}$$

$$\bar{\sigma}^{\mu} = (1, -\boldsymbol{\sigma}), \quad \sigma^{\mu} = (1, \boldsymbol{\sigma})$$

Remark: Choice of $\sqrt{}$ unambiguous. $p\cdot\sigma$ is a Hermitian matrix, diagonalised by unitary transformation.

$$p\cdot\sigma = u\begin{pmatrix} p^0 + |\boldsymbol{p}| & 0 \\ 0 & p^0 - |\boldsymbol{p}| \end{pmatrix}u^{-1}, \quad \sqrt{p\cdot\sigma} = u\begin{pmatrix} \sqrt{p^0 + |\boldsymbol{p}|} & 0 \\ 0 & \sqrt{p^0 + |\boldsymbol{p}|} \end{pmatrix}u^{-1}$$

(check by taking the square of $\sqrt{p\cdot\sigma}$).

Remark: Check equations of motion: $p\cdot\sigma\, p\cdot\,\bar{\sigma} = (E - \boldsymbol{p}\cdot\boldsymbol{\sigma})(E + \boldsymbol{p}\cdot\boldsymbol{\sigma}) = E^2 - p^2 = m^2$.

$$\not{p}' u = \begin{pmatrix} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} = \begin{pmatrix} p\cdot\bar{\sigma}\;\sqrt{p\cdot\sigma}\;\xi_s \\ p\cdot\sigma\,\sqrt{p\cdot\bar{\sigma}}\;\xi_s \end{pmatrix} = m\begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi_s \\ \sqrt{p\cdot\sigma}\,\xi_s \end{pmatrix} = m\,u(\boldsymbol{p})$$

$$(p\cdot\bar{\sigma}\;\sqrt{p\cdot\sigma})^2 = (m\,\sqrt{p\cdot\bar{\sigma}}\;)^2, \quad p\cdot\bar{\sigma}\,\sqrt{p\cdot\sigma}\;p\cdot\bar{\sigma}\,\sqrt{p\cdot\sigma} = m^2\,p\cdot\bar{\sigma}\,,$$

$$p\cdot\bar{\sigma}\;\;p\cdot\bar{\sigma}\;\left(\sqrt{p\cdot\sigma}\right)^2 =$$

Useful identities:

1. Orthogonality: $u_s^{\dagger}(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) = \xi_s^{\dagger}\Big(\left(\sqrt{p\cdot\sigma}\right)^2 + \left(\sqrt{p\cdot\bar{\sigma}}\;\right)^2\Big)\xi_{s'} = \xi_s^{\dagger}\,2\,p^0\,\xi_{s'} = 2\,E_p\,\delta_{ss'}$.

$$v_s^{\dagger}(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) = 2\,E_p\,\delta_{s s'}$$

$$u_s^{\dagger}(\boldsymbol{p})\,v_{s'}(-\boldsymbol{p}) = 0$$

2. $\bar{u}_s(\boldsymbol{p})\,u_{s'}(\boldsymbol{p}) = u_{L\,s}^{\dagger} u_{R\,s'} + u_{R\,s}^{\dagger} u_{L\,s'} = \xi_s^{\dagger}\big(\sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar{\sigma}} + \sqrt{p\cdot\bar{\sigma}}\,\sqrt{p\cdot\sigma}\big)\xi_{s'} = 2\,m\,\delta_{s s'}$.

$$\bar{v}_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) = -2\,m\,\delta_{s s'}$$

$$\bar{u}_s(\boldsymbol{p})\,v_{s'}(\boldsymbol{p}) = 0$$

3. Completenesslike: Note

$$\sum_s \xi_s \xi_s^{\dagger} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \mathbb{I}$$

$$\sum_s u_s(\boldsymbol{p})\,\bar{u}_s(\boldsymbol{p}) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}\Big(\xi_s^{\dagger}\sqrt{p\cdot\sigma}, \xi_s^{\dagger}\sqrt{p\cdot\bar{\sigma}}\,\Big) = \begin{pmatrix} \sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\Big(\;\sqrt{p\cdot\bar{\sigma}}\,, \quad \sqrt{p\cdot\sigma}\;\Big) =$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \not{p} + m$$

Similarly

$$\sum_s v_s(\boldsymbol{p})\,\bar{v}_s(\boldsymbol{p}) = \not{p} - m$$

Now we are ready to write down the Dirac field.

Expansion of Dirac field in solutions to equations:

$$\psi(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{\sqrt{2\,E_p}} \sum_s \left(a_s(\boldsymbol{p})\, u_s(\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}p\cdot x} + b_s^\dagger(\boldsymbol{p}) v_s(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}p\cdot x} \right)$$

$$\bar{\psi}\,(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{\sqrt{2 E_p}} \sum_s \left(a_s^\dagger(\boldsymbol{p}) \bar{u}_s(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}p\cdot x} + b_s(\boldsymbol{p})\, \bar{v}_s(\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}p\cdot x} \right)$$

Quantisation: $\mathcal{L} = \mathrm{i}\ \bar{\psi}\,(\not{\partial} - m)\psi = \mathrm{i}\ \psi^\dagger \psi + \cdots$. Compare $A = \int\ (p\dot{q} - H)\ \mathrm{d}t$. Conjugate momentum $\psi^\dagger$.

Quantisation:

$$\left[\psi_a(x), \psi_b^\dagger(y) \right]_+ \Big|_{x^0 = y^0} = \delta_{a\,b}\, \delta(\boldsymbol{x} - \boldsymbol{y}), \text{others} = 0$$

Equivalently:

$$\left[a_s(\boldsymbol{p}), a_{s'}^\dagger(\boldsymbol{p}') \right]_+ = (2\pi)^3\, \delta^3(\boldsymbol{p} - \boldsymbol{p}')\, \delta_{s\,s'},$$

$$\left[b_s(\boldsymbol{p}), b_{s'}^\dagger(\boldsymbol{p}) \right] = (2\pi)^3\, \delta^3(\boldsymbol{p} - \boldsymbol{p}')\, \delta_{s\,s'},$$

$$\text{others} = 0.$$

Energy operator

$$L = \int \mathrm{d}^3 x\, \mathcal{L}_0 = \int \mathrm{d}^3 x\, \mathrm{i}\, \psi^\dagger \dot{\psi} - \underbrace{\int \mathrm{d}^3 x\, \bar{\psi}\,(-\mathrm{i}\gamma^\kappa \nabla_\kappa + m)\psi}_{=H}$$

Using the equations of motion:

$$H = \int \mathrm{d}^3 x\, \mathrm{i}\, \psi^\dagger(x) \dot{\psi}(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2 E_p} \sum_s \sum_{s'} \left(a_s^\dagger(p) u_s^\dagger(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}E_p t} + b_s(-\boldsymbol{p}) v_s^\dagger(-\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}E_p t} \right) \cdot$$

$$\cdot\, E_p\big(a_{s'}(\boldsymbol{p}) u_{s'}(\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}E_p t} - b_{s'}(-\boldsymbol{p})\, v_{s'}(-\boldsymbol{p}) \mathrm{e}^{\mathrm{i}E_p t} \big) = \begin{bmatrix} \text{orthogonality relation} \\ \text{relation 1 above} \end{bmatrix} =$$

$$= (2\pi)^{-3} \int \mathrm{d}^3 p\, E_p \sum_s \left(a_s^\dagger(\boldsymbol{p})\, a_s(\boldsymbol{p}) - b_s(-\boldsymbol{p}) b_s(-\boldsymbol{p}) \right) =$$

$$= (2\pi)^{-3} \int \mathrm{d}^3 p\, E_p \sum_s \left(a_s^\dagger(\boldsymbol{p})\, a_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p}) b_s(\boldsymbol{p}) - (2\pi)^3 \delta^3(\boldsymbol{0}) \right)$$

Number operators: $a_s^\dagger(\boldsymbol{p})\, a_s(\boldsymbol{p}) + b_s^\dagger(\boldsymbol{p}) b_s(\boldsymbol{p}) = N_{e^-} + N_{e^+}$. Vacuum energy $-(2\pi)^3 \delta^3(\boldsymbol{0})$: negative for fermions.

Feynman propagator

$$S_F(x - y)_{a\,b} = \langle 0 | T \psi_a(x) \bar{\psi}_b(x) | 0 \rangle =$$

$$= \langle 0 | \big(\theta(x^0 - y^0)\, \psi_a(x)\, \bar{\psi}_b(y) - \theta\big(y^0 - x^0\big) \bar{\psi}_b(y) \psi_a(x) \big) | 0 \rangle$$

(You have to insert a fermion minus sign when switching order.)

$$\mathrm{i}\big(\not\partial_x - m\big)_{ac} S_F(x-y)_{cb} = \mathrm{i}\gamma^0_{ac}\langle 0|\delta(x^0-y^0)\underbrace{\big[\psi_a(x),\bar{\psi}_b(y)\big]_+}_{\delta_{cd}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})}|0\rangle\gamma^0_{db} = \mathrm{i}\delta_{ab}\delta^4(x-y)$$

This is the differential equation obeyed by this propagator. Fourier transform both sides:

$$S_F(x) = (2\pi)^{-4}\int \mathrm{d}^4p\,\tilde{S}(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\big(\not p - m\big)\tilde{S}(p) = \mathrm{i}$$

$$S_F(x) = (2\pi)^{-4}\int \mathrm{d}^4p\,\frac{\mathrm{i}}{\not p - m}\,\mathrm{e}^{-\mathrm{i}p\cdot x}$$

To make it well defined:

$$S_F = (2\pi)^{-4}\int \mathrm{d}^4p\,\frac{\mathrm{i}\big(\not p + m\big)}{p^2-m^2}\,\mathrm{e}^{-\mathrm{i}p\cdot x}$$

And now we have to make a pole prescription.

$$S_F = (2\pi)^{-4}\int \mathrm{d}^4p\,\frac{\mathrm{i}\big(\not p + m\big)}{p^2-m^2+\mathrm{i}\varepsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$\big(\not p - m\big)\tilde{S}(p) = \mathrm{i}.$

Closing p^0 integration contour in the lower half plane if $x^0>0$, in the upper half plane if $x^0<0$ gives

$$S_F(x) = (2\pi)^{-4}\cdot(-2\pi\mathrm{i})\int\frac{\mathrm{d}^3p}{2E_p}\,\mathrm{i}\big[\big(\not p+m\big)\mathrm{e}^{-\mathrm{i}p\cdot x}\theta(x^0) + \big(-\not p+m\big)\,\mathrm{e}^{\mathrm{i}p\cdot x}\theta(-x^0)\big]$$

Check

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|\psi_+(x)\bar{\psi}_-(y)|0\rangle = \langle 0|[\psi_+(x),\bar{\psi}_-(y)]_+|0\rangle =$$

$$= (2\pi)^{-3}\int\frac{\mathrm{d}^3p}{2E_p}\underbrace{\Big[\sum_s u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p})\Big]}_{=\not p+m}\mathrm{e}^{-\mathrm{i}p\cdot x}$$