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Last time: Free scalar field $\varphi(x)$. The theory is defined by the Lagrangian density, like this:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \varphi\, \partial_\mu \varphi - m^2 \varphi^2\right)$$

We have the Euler-Lagrange equations of motion: $\left(\partial^2 + m^2\right)\varphi = 0$. This is the Klein-Gordon equation. We solve the Klein–Gordon equation by expanding in Fourier components, and the solution was like this:

$$\varphi(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p}\left(\tilde{a}(\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i} p\cdot x} + \tilde{a}^*(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i} p\cdot x}\right),$$

where $\tilde{a}$ and $\tilde{a}^*$ become annihilation and creation operators after quantisation. The Hilbert space is a Fock space.

One can normalise $\tilde{a}$ differently. One can write

$$\tilde{a}^\dagger = \sqrt{2E_p}\, a^\dagger(\boldsymbol{p}).$$

The a is the one Peskin-Schroeder is using, while we have been using $\tilde{a}$. $a^\dagger$ is the more natural creation operator, but $\tilde{a}^\dagger$ has nicer Lorentz transformation properties. The reason is, roughly, that φ is a scalar field, which transforms simply under a Lorentz transformation. See Peskin-Schroeder equation 2.38 about this.

Now $P^\mu = (H, \boldsymbol{P})$ are the energy and momentum operators — operators with capital letters, to distinguish from the numbers p^μ. Obtained by Noether's procedure. Infinitesimal symmetry translation yields a conserved current j^μ. This leads to a conserved charge

$$Q = \int \mathrm{d}^3 x\, j^0,$$

see Peskin-Schroeder. For energy and momentum, the symmetry is translation symmetry: $\delta\varphi = \varphi(x+\varepsilon) - \varphi(x) = \varepsilon^\nu \partial_\nu \varphi$, leads to a conserved current $T^{\mu\nu} = \partial^\mu\varphi\, \partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$. **Show:** $\partial_\mu T^{\mu\nu} = 0$ using the equations!

Conserved charges

$$\int \mathrm{d}^3 x\, \frac{1}{2}\left(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right) = H \text{ (operator, when quantised)}$$

$$\int \mathrm{d}^3 x\, \pi \nabla\varphi = -\boldsymbol{P} \text{ (operator)}$$

Check these equations:

$$[\mathrm{i}H, \varphi] = \pi = \dot{\varphi}$$

$$[\mathrm{i}H, \pi] = \left(\nabla^2 - m^2\right)\varphi = \dot{\pi} \quad \text{(by equation of motion)}$$

$$[\mathrm{i}\boldsymbol{P}, \varphi(x)] = -\nabla\varphi(x)$$

It is a simple calculation, but it might not be simple the first time you do it. The more difficult you find it, the more important it it that you do it.

Expressions of H and $\boldsymbol{P}$ in terms of creation and annihilation operators:

$$\boldsymbol{P} = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2\,E_p} \boldsymbol{p}\, \tilde{a}^\dagger(\boldsymbol{p})\, \tilde{a}(\boldsymbol{p})$$

$$H = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2\,E_p} \frac{1}{2}\big(\tilde{a}^\dagger(\boldsymbol{p})\tilde{a}(\boldsymbol{p}) + \tilde{a}(\boldsymbol{p})\, \tilde{a}^\dagger(\boldsymbol{p})\big) E_p =$$

$$= \underbrace{(2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p}}_{\text{number operator}} \left(\tilde{a}^\dagger(\boldsymbol{p})\tilde{a}(\boldsymbol{p}) + \frac{1}{2}(2\pi)^3\, 2E_p\, \delta^3(0) \right) E_p$$

$\delta^3(0)$ is infinite! And integrating over all momenta, it becomes even more infinite. We have infinite vacuum energy. Usually people just ignore this, since such an energy would not be observable. We can only measure energy differences, not total energies. (Well, in general relativity, all energy matters, and infinite energy would give an infinite curvature of spacetime. Gravitation has to be excluded for things to work.)

$$\left[\mathrm{i}\, H^{\mathrm{op}}, \tilde{a}^\dagger(\boldsymbol{p})\right] = E_p\, \tilde{a}^\dagger(\boldsymbol{p})$$

$$[\mathrm{i}\, H^{\mathrm{op}}, \tilde{a}(\boldsymbol{p})] = -\, E_p \tilde{a}(\boldsymbol{p})$$

$$\left[\mathrm{i}\, \boldsymbol{P}^{\mathrm{op}}, \tilde{a}^\dagger(\boldsymbol{p})\right] = \boldsymbol{p}\, \tilde{a}^\dagger(\boldsymbol{p})$$

$$[\mathrm{i}\, \boldsymbol{P}^{\mathrm{op}}, \tilde{a}(\boldsymbol{p})] = -\, \boldsymbol{p}\, \tilde{a}(\boldsymbol{p})$$

Check all these.

$\tilde{a}^\dagger(\boldsymbol{p})/\tilde{a}(\boldsymbol{p})$ increase/decrease energy by E_p, momentum by $\boldsymbol{p}$, as creation and annihilation operators should.

Green functions (propagators) of Klein-Gordon operator $\partial^\mu \partial_\mu + m^2$. One calls the propagator $D(x-y)$. $D(x-y)$ of the Klein-Gordon operator $\partial^2 + m^2$ satisfies the differential equation:

$$\left(\partial^2 + m^2\right) D(x-y) = -\,\mathrm{i}\, \delta^4(x-y).$$

This is solved by a Fourier transformation. Then one has

$$\delta^4(x-y) = (2\pi)^{-4} \int \mathrm{d}^4 p\, \mathrm{e}^{-\mathrm{i} p\cdot(x-y)}$$

$$D(x-y) = (2\pi)^{-4} \int \mathrm{d}^4 p\, \tilde{D}(p)\, \mathrm{e}^{-\mathrm{i} p\cdot(x-y)}$$

$$\left(-\, p^2 + m^2\right) \tilde{D}(p) = -\,\mathrm{i}, \quad \tilde{D}(p) = \frac{\mathrm{i}}{p^2 - m^2}$$

Problem: The denominator has a zero. (We are integrating over all p.)

$$p^2 - m^2 = \left(p^0\right)^2 - \boldsymbol{p}^2 - m^2 = \left(p^0\right)^2 - E_p^2$$

In the complex p^0 plane. We integrate over the real axis. Two poles, at E_p and $-E_p$.

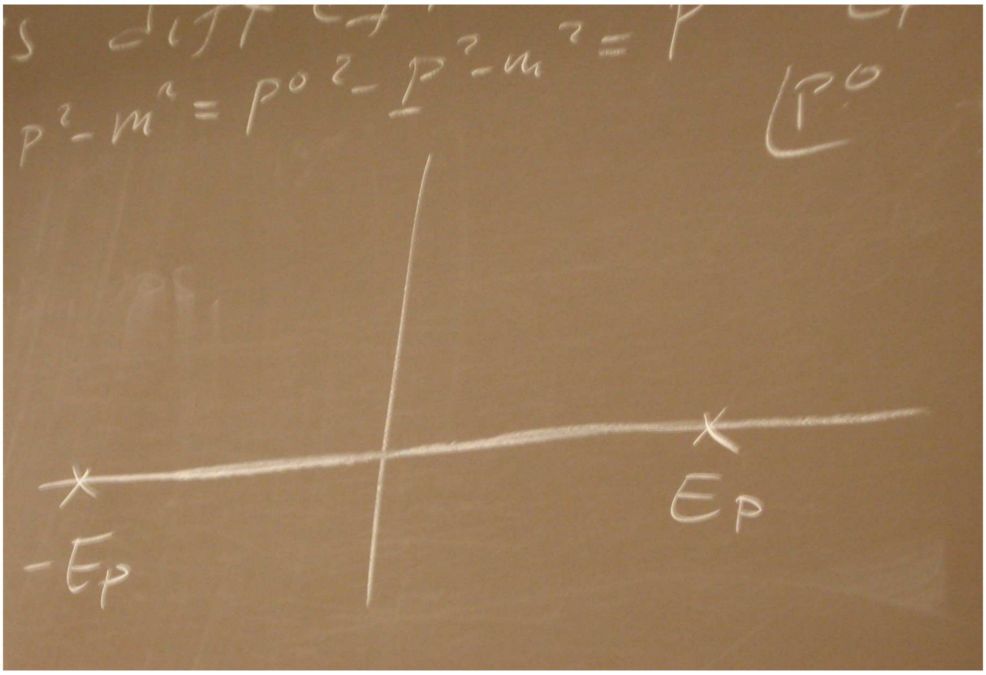

Figure 1. There are two poles in the complex p^0 plane, at E_p and at $-E_p$.

The way to handle them is to deform the integration contour so that we go around these poles. You can do it in different ways and get different Green's functions.

Insert in integral, perform integral. p^0 integration requires pole prescription. Different possibilities $\Rightarrow$ different Green's functions. The most important is the Feynman propagator: you go below the poles to the left and above those to the right.

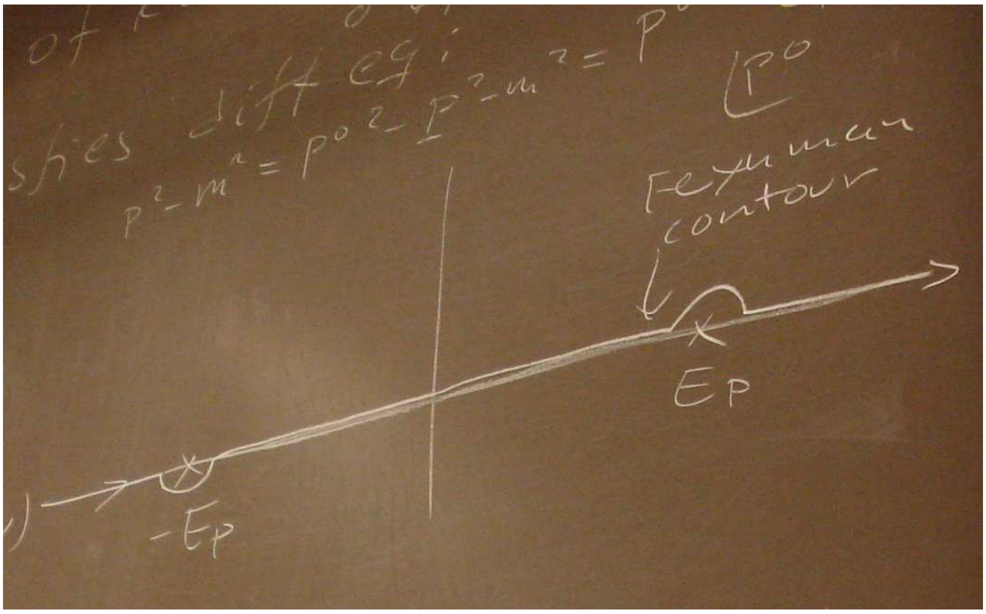

Figure 2. The Feynman contour, going below and above the poles, respectively.

Equivalently, move the poles slightly:

$$\tilde{D}_F(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

$$D_F(x) = (2\pi)^{-4} \int \mathrm{d}^4 p \, \frac{\mathrm{i}}{p^2 - m^2} \, \mathrm{e}^{-\mathrm{i} p \cdot x}$$

If $x^0 > 0$, $\mathrm{e}^{-\mathrm{i} p^0 x^0} \to 0$ as $\operatorname{Im} p^0 \to -\infty$, then close the contour in the lower half plane.

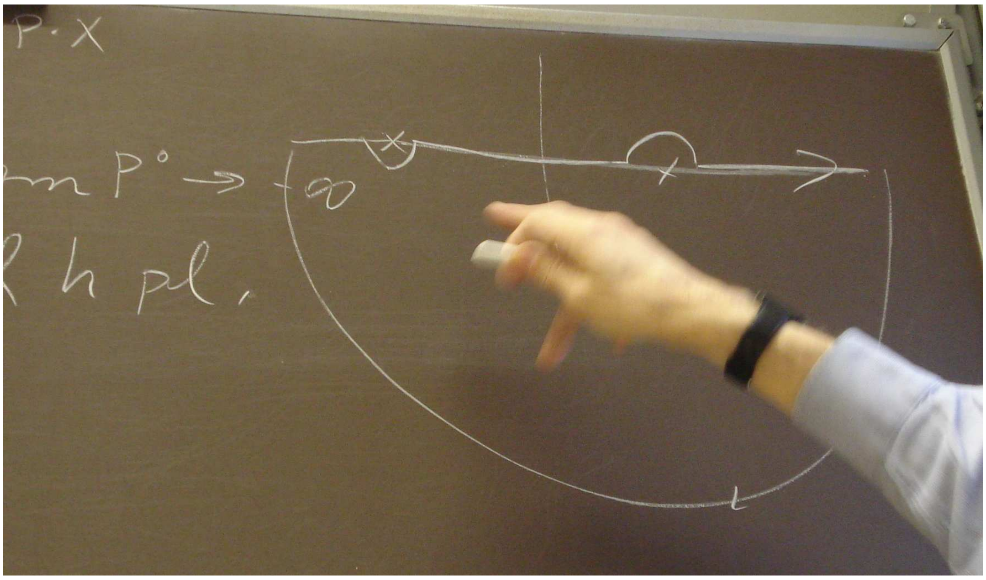

Figure 3. If $x^0 > 0$, close the contour in the lower half plane.

If $x^0 < 0$ we close the contour in the upper half plane instead.

$$D_F(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p} \left(\theta(x^0) \, \mathrm{e}^{-\mathrm{i} p \cdot x} + \theta(-x^0) \, \mathrm{e}^{\mathrm{i} p \cdot x} \right) \Big|_{p^0 = E_p}$$

$$D_F(x - y) = \langle 0 | \big(\theta(x^0 - y^0) \varphi(x) \varphi(y) + \varphi(y^0 - x^0) \varphi(y) \varphi(x) | 0 \rangle \equiv \langle 0 | T(\varphi(x) \varphi(y)) | 0 \rangle$$

Proof:

$$\varphi(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p} \big(\tilde{a}(\boldsymbol{p}) \, \mathrm{e}^{-\mathrm{i} p \cdot x} + \tilde{a}^\dagger(\boldsymbol{p}) \, \mathrm{e}^{\mathrm{i} p \cdot x} \big) = \varphi_+(x) + \varphi_-(x)$$

$$\Rightarrow \varphi_+(x) | 0 \rangle = 0, \quad \langle 0 | \varphi_-(x) = 0$$

$$\Rightarrow \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \langle 0 | \theta(x^0 - y^0) [\varphi_+(x), \varphi_-(y)] + \theta(x^0 - y^0) [\varphi_+(y), \varphi_-(x)] | 0 \rangle =$$

$$= \theta(x^0 - y^0) [\varphi_+(x), \varphi_-(y)] + \theta(x^0 - y^0) [\varphi_+(y), \varphi_-(x)] \underbrace{\langle 0 | 0 \rangle}_{= 1}$$

$$[\varphi_+(x), \varphi_-(y)] = (2\pi)^{-6} \int \frac{\mathrm{d}^3 p}{2E_p} \int \frac{\mathrm{d}^3 p'}{2E_{p'}} \mathrm{e}^{-\mathrm{i}(p \cdot x - p' \cdot y)} \big[\tilde{a}(\boldsymbol{p}), \, \tilde{a}^\dagger(\boldsymbol{p}') \big] =$$

$$\left[\quad \big[\tilde{a}(\boldsymbol{p}), \, \tilde{a}^\dagger(\boldsymbol{p}') \big] = (2\pi)^3 \cdot 2E_{p'} \, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \right]$$

$$= (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2E_p} \, \mathrm{e}^{-\mathrm{i}(p \cdot x - p \cdot y)}$$

Showing directly that

$$\left(\partial^2+m^2\right)\langle 0|T\varphi(x)\varphi(y)|0\rangle=-\,\mathrm{i}\,\delta^4(x-y)$$

$$\left(\partial_0\partial_0-\nabla^2+m^2\right)\langle 0|\left(\theta(x^0-y^0)\varphi(x)\varphi(y)+\theta(y^0-x^0)\varphi(y)\varphi(x)\right)|0\rangle$$

First ∂_0 acting on θ functions give

$$\langle 0|\underbrace{\delta(x^0-y^0)[\varphi(x),\varphi(y)]}_{=0}\ldots|0\rangle$$

When the second time derivative acts on θ:

$$\langle 0|\delta(x^0-y^0)[\pi(x),\varphi(y)]|0\rangle=-\,\mathrm{i}\delta^4(x-y).$$

Interpretation of this: It is the probability amplitude for particle created at y to occur at x if $x^0>y^0$, and if $y^0>x^0$ it is the amplitude for a particle created at x to occur at y. However, these probabilities are not zero outside the light cone. We have to worry about causality. But causality is actually a weaker condition than that the wave functions are zero outside the light cone. Causality only requires that non-causally connected observables don't affect each other, i.e. $[O(x),O(y)]=0$ if $(x-y)^2<0$.

Example in Klein-Gordon theory. Suppose $(x-y)^2<0$:

$$[\varphi(x),\varphi(y)]=[\varphi_+(x),\varphi_-(y)]+[\varphi_-(x),\varphi_+(y)]=$$

$$=(2\pi)^3\int\frac{\mathrm{d}^3p}{2E_p}\left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right)$$

If $x^0=y^0$, we see that this is zero. If $x^0\neq y^0$ (but still with spacelike separation), we can make a Lorentz transformation to get $x^0=y^0$. The above expression has nice Lorentz transformation properties. This is the way in which quantum field theory saves causality.

Box normalisation:

Space $\rightarrow$ box of length L with periodic boundary conditions.

Momenta are discrete $\boldsymbol{p}=\frac{2\pi}{L}(n_1,n_2,n_3)$.

$$\varphi(x)=L^{-3/2}\sum_p\frac{1}{\sqrt{2E_p}}\left(a(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x}+a^\dagger(\boldsymbol{p})\,\mathrm{e}^{\mathrm{i}p\cdot x}\right)\Big|_{p_0=\sqrt{\boldsymbol{p}^2+m^2}=E_p}$$

$$\pi(x)=\dot{\varphi}(x)$$

$$[\pi(x),\varphi(x)]=-\,\mathrm{i}\delta^3(x-y)$$

$$\Rightarrow\left[a(\boldsymbol{p}),a^\dagger(\boldsymbol{p})\right]=\delta_{\boldsymbol{p}\boldsymbol{p}'}$$

$$\boldsymbol{P}^{\mathrm{op}}=\sum_{\boldsymbol{p}}a^\dagger(p)a(p)\boldsymbol{p}=\sum_{\boldsymbol{p}}N_{\boldsymbol{p}}^{\mathrm{op}}\boldsymbol{p}$$

$$H^{\mathrm{op}}=\sum_p E_p\left(\frac{1}{2}+N_p^{\mathrm{op}}\right)$$