# 2009–01–21

The course home page is found at `http://fy.chalmers.se/~tfeps/qft.dir/qft.html`

Adding one lecture: Monday 08–10.

In this course: elementary particle cross sections.

Motivation: Why study quantum field theory? Because relativistic quantum mechanics implies quantum field theory. The real world is relativistic, and quantum mechanical, so to describe it you need quantum field theory. Whenever both relativity and quantum mechanics are needed to describe the world, you need quantum field theory. This is because:

1. Particles can be created and destroyed, e.g. $\gamma+\gamma\rightarrow e^-+e^+$. There is no general conservation law for particle number. You can create matter particles from pure light. Elementary particle physics is one area where quantum field theory is needed, and is the focus of this course.

Quantum field theory is useful in more contexts where particles are created and destroyed. That happens some times in solid state theory. At such low energies you can't produce electrons, but in descriptions without conserved particle number, quantum field theory finds its uses nevertheless.

2. There is a causality problem in quantum mechanics. A wave function that is local at time 0 is usually spread all over space at time $t>0$. Take for instance

$$\psi(\boldsymbol{x},0)=(2\pi)^3\,\delta^3(\boldsymbol{x}-\boldsymbol{x}_0)=\int \mathrm{d}^3p\,\exp(\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_0))$$

Momentum is conserved, so plane waves such as those superimposed in the integral evolve like

$$\int \mathrm{d}^3p\,\exp(\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)-\mathrm{i}\,E_p\,t)=\psi(\boldsymbol{x},t)$$

This is not localised in any finite region of space. Non-relativistically, this is not a problem, but in relativistic physics we should like it to be constrained to a light cone. Physical effects must not propagate faster than the speed of light, $c$. (In quantum field theory we often use units such that $c=1$ and $\hbar=1$.)

Quantum field theory solves this problem. It has a feature called microcausality, which we will comment on later on.

### General features of Quantum Field Theory

One has one quantum field for each particle type. Example particle: the Higgs particle (which has not been found yet). It has spin zero. It is described by a scalar field $\varphi(x)$.

The electron and positron have spin 1/2. We use the field $\psi_a(x)$. The index $a=1,\ldots,4$, but it is not a Lorentz index. We also have $\psi_a^\dagger(x)$.

You can think of the field as a classical field, but in the quantum theory it is a quantum field — an operator. The field becomes an operator.

As a classical field $\psi_a(x)$ is a complex valued field.

The photon has spin one, and the field is $A_\mu(x)$ — the classical field is the vector potential of electromagnetism.

| Example particle | Spin | Field |
|---|---|---|
| Higgs particle | 0 | $\varphi(x)$ |
| Electron+positron | 1/2 | $\psi_a(x), \psi_a^\dagger(x)$ |
| Photon | 1 | $A_\mu(x)$ |

**Table 1.**

Calculations in quantum field theory is done using perturbation theory.

To define a quantum field theory, you have to define the fields you have, and the Lagrangian density. The action of a field theory is written as an integral of the Lagrangian density $\mathcal{L}$:

$$\text{action} = \int \mathrm{d}^4x\, \mathcal{L}(x).$$

The Lagrangian density can be separated into two terms,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

where $\mathcal{L}_0$ is the free field Lagrangian density, describing free particles. It is quadratic in fields, which gives linear equations of motion. Such linear equations are exactly solvable (we will see how they describe free particles). $\mathcal{L}_I$ describes interactions between the particles in $\mathcal{L}_0$. You do perturbation theory in terms of the effects of $\mathcal{L}_I$, where the terms in the expansion are related to Feynman diagrams.

So how can such a quadratic Lagrangian $\mathcal{L}_0$ describe free particles? (Compare with the harmonic oscillator. $E_n = \left(\frac{1}{2} + n\right) \hbar\,\omega$, $n \in \mathbb{N} = \text{number of quanta}$.)

**Free scalar field** $\varphi(x)$. This is the simplest free field.

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\varphi(x)\,\partial^\mu\varphi(x) - \frac{1}{2}\,m^2\varphi(x)^2$$

$$\partial_\mu\varphi \equiv \varphi_{,\mu} \equiv \frac{\partial\varphi}{\partial x^\mu}$$

Indices are raised and lowered using matrices $\eta^{\mu\nu}$, $\eta_{\mu\nu}$, both equal to $\mathrm{diag}(1, -1, -1, -1)$.

$$\partial^\mu = \eta^{\mu\nu}\,\frac{\partial}{\partial x^\nu} \quad \text{(remember the summation convention!)}$$

Equation:

$$\left(\partial^2 + m^2\right)\varphi(x) = 0$$

This can be solved by a Fourier transformation:

$$\varphi(x) = (2\pi)^{-3} \int \mathrm{d}^4p\, \tilde{\varphi}(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x}$$

The equation of motion implies

$$\tilde{\varphi}(p) = \delta(p^2 - m^2)\tilde{a}(p)$$

Identity

$$\delta(p^2 - m^2) = \frac{1}{2E_p}(\delta(p_0 - E_p) + \delta(p_0 + E_p)), \quad E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

This identity comes from the factorisation:

$$0 = p_0^2 - \boldsymbol{p}^2 - m^2 = \left(p_0 - \sqrt{\boldsymbol{p}^2 + m^2}\right)\left(p_0 + \sqrt{\boldsymbol{p}^2 + m^2}\right), \quad \delta(a(x)\,x) = \frac{1}{a(0)}\,\delta(x), \text{for nice functions}$$

(without zeroes) $a(x)$. If $a(x)$ has zeroes we get additional terms, which was the case above. Note that $p^2 = p_\mu p^\mu = (p_0)^2 - \boldsymbol{p}^2$, where $\boldsymbol{p}$ is the three-dimensional momentum. Defining $E_p \equiv \sqrt{\boldsymbol{p}^2 + m^2}$, so we get

$$\delta(p^2 - m^2) = \delta\big(p_0^2 - \boldsymbol{p}^2 - m^2\big) = \delta\big(p_0^2 - E_p^2\big) = \delta((p_0 - E_p)(p_0 + E_p)) =$$

$$= \frac{1}{2E_p}(\delta(p_0 - E_p) + \delta(p_0 + E_p))$$

The first of these two deltas is the positive energy mass shell, while the second is the negative energy mass shell.

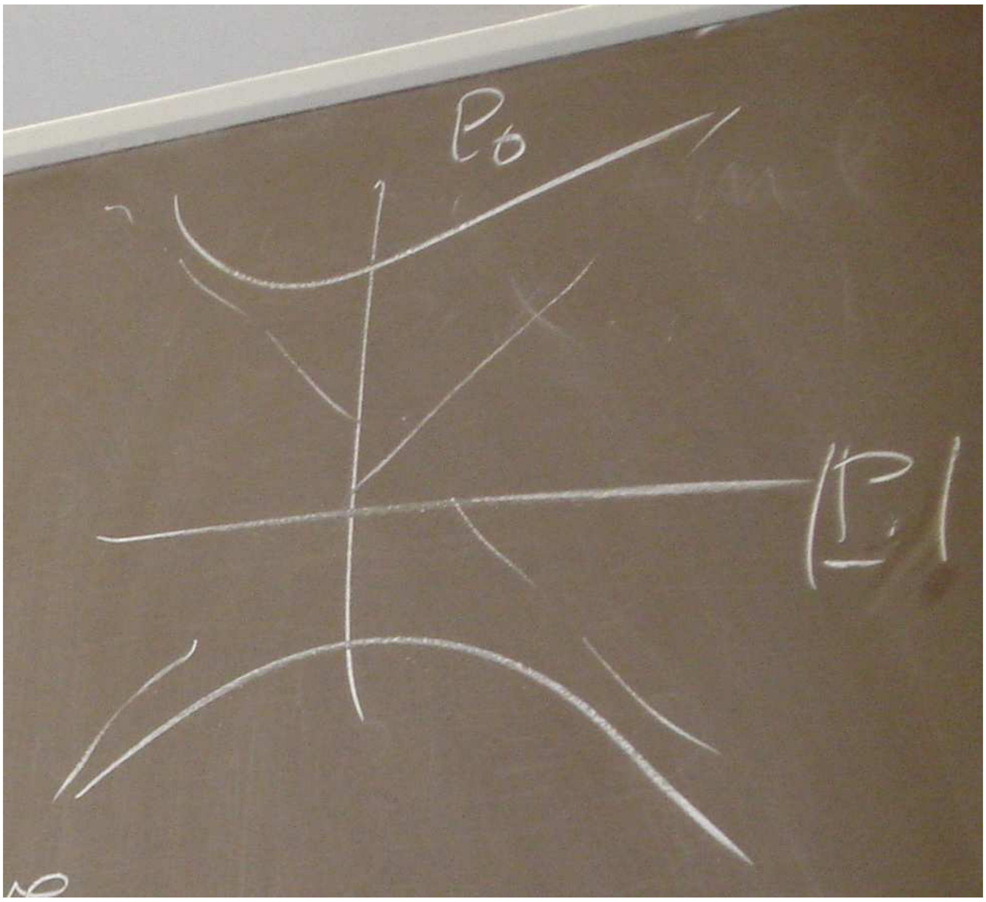


**Figure 1.** Positive and negative energy mass shell.

$$\varphi(x) = (2\pi)^{-3} \int \mathrm{d}^4p\, \tilde{\varphi}(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x} = (2\pi)^{-3} \int \mathrm{d}^4p\, \delta(p^2 - m^2) \tilde{a}(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x} =$$

$$= (2\pi)^{-3} \int \frac{\mathrm{d}^3p}{2E_p} \big(\tilde{a}(\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}p\cdot x} + \tilde{a}(\boldsymbol{p})^*\, \mathrm{e}^{\mathrm{i}p\cdot x}\big) \quad \text{with } p_0 = \sqrt{\boldsymbol{p}^2 + m^2},$$

where we use that $\varphi(x)$ is real, so $\tilde{a}(p)^* = \tilde{a}(-p)$. One defines $\tilde{a}(\boldsymbol{p})$ (a function of the three-momentum) by

$$\tilde{a}(\boldsymbol{p}) = \tilde{a}(p_0, \boldsymbol{p}) \text{ with } p_0 = \sqrt{\boldsymbol{p}^2 + m^2} = E_p.$$

**Quantisation.** We quantise by using the correspondence between the Poisson bracket $[\,*\,,*\,]_{\mathrm{PB}}$ and the commutator $[\,*\,,*\,]$. We will need the conjugate momentum:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x),$$

$$\pi(x) = \mathrm{i}(2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{2\,E_p} \cdot E_p\big(\tilde{a}(\boldsymbol{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} - \tilde{a}(\boldsymbol{p})^*\,\mathrm{e}^{\mathrm{i}p\cdot x}\big)$$

The correspondence relation is

$$[\,*\,,*\,]_{\mathrm{PB}} \to \frac{1}{\mathrm{i}\hbar}[\,*\,,*\,]$$

where $[\,*\,,*\,]$ is the quantum mechanical commutator. (We set $\hbar = 1$.) $\varphi$, $\pi$, $\tilde{a}$, $\tilde{a}^*$ become operators satisfying commutation relations:

$$[\pi(x), \varphi(y)]\Big|_{x^0=y^0} = -\,\mathrm{i}\,\delta^3(\boldsymbol{x} - \boldsymbol{y})$$

$$[\varphi(x), \varphi(y)]\Big|_{x^0=y^0} = 0 = [\pi(x), \pi(y)]\Big|_{x^0=y^0}$$

$$\Rightarrow \begin{cases} \big[\tilde{a}(\boldsymbol{p}), \tilde{a}(\boldsymbol{p}')^\dagger\big] = (2\pi)^3\,2E_p\,\delta^3(\boldsymbol{p} - \boldsymbol{p}'), \\ [\tilde{a}(\boldsymbol{p}), \tilde{a}(\boldsymbol{p}')] = 0 = \big[\tilde{\boldsymbol{a}}(\boldsymbol{p})^\dagger, \tilde{a}(\boldsymbol{p}')^\dagger\big] = 0 \end{cases}$$

Show this, using $\int\ \mathrm{d}^3 p\,\exp(\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x} - \boldsymbol{y})) = (2\pi)^3\,\delta^3(\boldsymbol{x} - \boldsymbol{y})$ and expressions for $\pi(x)$, $\varphi(x)$.

$\tilde{a}(\boldsymbol{p})$ and $\tilde{a}(\boldsymbol{p})^\dagger$ are annihilation and creation operators, respectively, for particles of momentum $\boldsymbol{p}$. If you have seen these operators and are uncomfortable with the $\delta^3(\boldsymbol{p} - \boldsymbol{p}')$, one can introduce a large box, making momentum discrete: $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = \delta_{\boldsymbol{p}\boldsymbol{p}'}$. If we define $\tilde{a}$ differently, we can remove the $2E_p$ in $\big[\tilde{a}(\boldsymbol{p}), \tilde{a}(\boldsymbol{p}')^\dagger\big]$, so $\tilde{a}(\boldsymbol{p})$ and $\tilde{a}(\boldsymbol{p})^\dagger$ are unconventionally normalised annihilation and creation operators.

One can define a Hilbert space that contains a vacuum state $|0\rangle$ and has the property $\tilde{a}(p)|0\rangle = 0$: any annihilation operator acting on the vacuum state is zero. The vacuum has $\langle 0|0\rangle = 1$. A one particle state is $|\boldsymbol{p}\rangle \equiv \tilde{a}(\boldsymbol{p})^\dagger|0\rangle$. A two-particle state is $|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = \tilde{a}^\dagger(\boldsymbol{p}_1)\,\tilde{a}^\dagger(\boldsymbol{p}_2)|0\rangle$. Such a Hilbert space is called a Fock space.

$$\langle p|p'\rangle = \big(\tilde{a}^\dagger(\boldsymbol{p})|0\rangle\big)^\dagger \tilde{a}^\dagger(\boldsymbol{p}')|0\rangle = \langle 0|\tilde{a}(\boldsymbol{p})\,\tilde{a}^\dagger(\boldsymbol{p}')|0\rangle = \langle 0|\big[\tilde{a}(\boldsymbol{p}), \tilde{a}^\dagger(\boldsymbol{p}')\big]|0\rangle =$$

$$= 2E_p(2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}').$$