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$$A\,u_{x\,x} + 2\,B\,u_{x\,y} + C\,u_{y\,y} + \text{lägre termer} = F$$

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y)$$

$$\Rightarrow \tilde{A}\,u_{\xi\xi} + 2\tilde{B}\,u_{\xi\eta} + \tilde{C}u_{\eta\eta} + \text{lägre termer} = F$$

$$\tilde{A} = A\,\varphi_x\varphi_x + 2\,B\,\varphi_x\varphi_y + C\,\varphi_y\varphi_y$$

$$\tilde{B} = A\,\varphi_x\psi_x + C\,\varphi_y\,\psi_y + B\,(\varphi_x\psi_y + \varphi_y\psi_x)$$

$$\tilde{C} = A\psi_x\psi_x + 2\,B\psi_x\psi_y + C\,\psi_y\psi_y$$

$$B^2 - A\,C\begin{cases} > 0: & \text{hyperbolisk} \\ = 0: & \text{parabolisk} \\ < 0: & \text{elliptisk} \end{cases}$$

1. Hyperbolisk: $B^2 - A\,C > 0$.

Vi vill försöka annullera $\tilde{A}$ och $\tilde{C}$.

$$A\,\varphi_x\varphi_x + 2\,B\,\varphi_x\,\varphi_y + C\,\varphi_y\,\varphi_y = 0$$

Delar med $\varphi_x \cdot \varphi_x$:

$$A + 2\,B\,\frac{\varphi_y}{\varphi_x} + C\left(\frac{\varphi_y}{\varphi_x}\right)^2 = 0$$

$$\frac{\varphi_y}{\varphi_x} = \frac{-B \pm \sqrt{B^2 - A\,C}}{C}$$

$$C\,\varphi_y + \left(B \pm \sqrt{B^2 - A\,C}\right)\varphi_x = 0$$

För $a\varphi_x + b\,\varphi_y = 0$ betraktar man den karakteristiska ekvationen $a\,\mathrm{d}y = b\,\mathrm{d}x$. I detta fallet:

$$\begin{cases} a &=& B \pm \sqrt{B^2 - A\,C} \\ b &=& C \end{cases}$$

Vi tar φ, ψ som lösningar till karakteristiska ekvationen.

$$2\tilde{B}u_{\xi 2} + \text{lägre termer} = F$$

(kanonisk form).

2. Parabolisk typ: $B^2 - A\,C = 0$. Vi ska annullera $\tilde{A}$ och $\tilde{B}$.

$$A\varphi_x\varphi_x + 2\,B\,\varphi_x\varphi_y + C\,\varphi_y\varphi_y = 0$$

$$\frac{\varphi_y}{\varphi_x} = \frac{-B \pm \sqrt{0}}{C}$$

$$B\,\varphi_x + C\,\varphi_y = 0$$

$$\xi = \varphi(x, y)$$

Som η tar vi $\eta = \psi(x, y)$ som är *oberoende* av $\varphi(x, y)$.

$$B\psi_x + C\,\psi_y \neq 0 \quad \text{eller} \quad \begin{vmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{vmatrix} \neq 0.$$

3. Elliptisk typ: $B^2 - A\,C < 0$.

Skriver karakteristiska ekvationen:

$$A\,\varphi_x\varphi_x + 2\,B\,\varphi_x\,\varphi_y + C\,\varphi_y\,\varphi_y = 0.$$

$$\frac{\varphi_y}{\varphi_x} = \frac{-B \pm \sqrt{B^2 - A\,C}}{C}$$

$$C\,\varphi_y + \underbrace{\left(B \pm \sqrt{B^2 - A\,C}\right)}_{\text{komplexa koefficienter}}\varphi_x = 0??$$

Skriver $C\,\mathrm{d}x = \left(B + \sqrt{B^2 - A\,C}\right)\mathrm{d}y$. $\varphi(x, y) = \gamma$ är allmänna lösningen, $\varphi = \varphi_1 + \mathrm{i}\varphi_2$.

$$\xi = \varphi_1(x, y); \quad \eta = \varphi_2(x, y)$$

$\Rightarrow \tilde{B} = 0$; $\tilde{A} = \tilde{C}$.

Exempel:

$$u_{x\,x} - 2\cos x\,u_{x\,y} - (3 + \sin^2 x)u_{y\,y} - y\,u_y = 0$$

$$\begin{cases} A &= 1 \\ B &= -\cos x \\ C &= -3 - \sin^2 x \end{cases}$$

$$B^2 - A\,C = \cos^2 x + 3 + \sin^2 x = 4 > 0$$

$$\varphi_x\,\varphi_x - 2\cos x \cdot \varphi_x\varphi_y - (3 + \sin^2 x)\varphi_y\varphi_y = 0$$

Delar med $\varphi_y\varphi_y$:

$$\left(\frac{\varphi_x}{\varphi_y}\right)^2 - 2\cos x \cdot \frac{\varphi_x}{\varphi_y} - 3 - \sin^2 x = 0$$

$$\frac{\varphi_x}{\varphi_y} = \cos x \pm \sqrt{4} = \cos x \pm 2$$

$$\varphi_x + (-\cos x \pm 2)\varphi_y = 0$$

Karakteristiska ekvationen: $\mathrm{d}y = (-\cos x \pm 2)\mathrm{d}x$.

$$\int \mathrm{d}y = \int (-\cos x \pm 2)\mathrm{d}x$$

$$y = -\sin x \pm 2\,x + \gamma$$

$$y + \sin x + 2x = \gamma$$

$$y + \sin x - 2\,x = \gamma$$

$$\xi = \varphi(x, y) = y + \sin x + 2x$$

$$\eta = \psi(x, y) = y + \sin x - 2x$$

$\tilde{A} = \tilde{C} = 0$. Vill beräkna $\tilde{B}$:

$$u_x = u_\xi \varphi_x + u_\eta \psi_x = u_\xi(\cos x + 2) + u_\eta(\cos x - 2)$$

$$u_{xx} = (u_\xi(\cos x + 2))_x + (u_\eta(\cos x - 2))_x =$$

$$= u_{\xi x}(\cos x + 2) + u_\xi(-\sin x) + u_{\eta x}(\cos x - 2) + u_\eta(-\sin x) =$$

$$u_{\xi\xi}(\cos x + 2)\varphi_x + u_{\xi\eta}(\cos x + 2)\psi_x - u_\xi \sin x + u_{\eta\xi}(\cos x - 2)\varphi_x + u_{\eta\eta}(\cos x - 2)\psi_x + u_\eta(-\sin x)$$

Exempel:

$$y^2 u_{xx} + 2xy u_{xy} + x^2 u_{yy} = 0$$

$$\begin{cases} A &= y^2 \\ B &= xy \\ C &= x^2 \end{cases}$$

$$B^2 - AC = (xy)^2 - x^2 y^2 = 0$$

Karakteristiska ekvationen:

$$y^2 \varphi_x \varphi_x + 2xy \varphi_x \varphi_y + x^2 \varphi_y \varphi_y = 0$$

$$y^2 + 2xy \frac{\varphi_y}{\varphi_x} + x^2 \left(\frac{\varphi_y}{\varphi_x}\right)^2$$

$$\frac{\varphi_y}{\varphi_x} = \frac{-xy \pm \sqrt{(xy)^2 - x^2 y^2}}{x^2}; \quad \frac{\varphi_y}{\varphi_x} = -\frac{y}{x}$$

$$x \varphi_y + y \varphi_x = 0$$

Karakteristiska ekvationen: $y\,\mathrm{d}y = x\,\mathrm{d}x$

$$\int y\,\mathrm{d}y = \int x\,\mathrm{d}x, \quad \frac{y^2}{2} = \frac{x^2}{2} + \frac{\gamma}{2}$$

$$y^2 - x^2 = \gamma$$

Svar: $\xi = \varphi(x, y) = y^2 - x^2$. $\eta = ?$ Vi kan ta nästan vad som helst, bara det är oberoende. $\eta = x$.

Exempel:

$$(1 + x^2) u_{xx} + (1 + y^2) u_{yy} = 0$$

$$\begin{cases} A &= 1 + x^2 \\ B &= 0 \\ C &= 1 + y^2 \end{cases}$$

$$B^2 - AC = -(1 + x^2)(1 + y^2)$$

Elliptisk.

$$(1+x^2)\varphi_x\varphi_x + \left(1+y^2\right)\varphi_y\varphi_y = 0$$

$$(1+x^2) + \left(1+y^2\right)\left(\frac{\varphi_y}{\varphi_x}\right)^2 = 0$$

$$\frac{\varphi_y}{\varphi_x} = \left(-\frac{1+x^2}{1+y^2}\right)^{\frac{1}{2}} = \mathrm{i}\frac{\sqrt{1+x^2}}{\sqrt{1+y^2}}$$

$$\varphi_y\left(1+y^2\right)^{\frac{1}{2}} - \mathrm{i}\sqrt{1+x^2}\,\varphi_x = 0$$

$$-\mathrm{i}\sqrt{1+x^2}\,\mathrm{d}y = \sqrt{1+y^2}\,\mathrm{d}x$$

$$\frac{-\mathrm{i}\,\mathrm{d}y}{\sqrt{1+y^2}} = \frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

$$-\mathrm{i}\int\frac{\mathrm{d}y}{\sqrt{1+y^2}} = \int\frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

$$-\mathrm{i}\ln(y+\sqrt{1+y^2}) = \ln(x+\sqrt{1+x^2}) + \gamma$$

$$\underbrace{\ln\left(x+\sqrt{1+x^2}\right)}_{=\varphi(x,y)} + \mathrm{i}\underbrace{\ln\left(y+\sqrt{1+y^2}\right)}_{=\psi(x,y)} = \gamma$$

EXEMPEL:

$$\sin^2 x\, u_{xx} - 2\,y\sin x\cdot u_{xy} + y^2\,u_{yy} = 0$$

$$\begin{cases} A &=& \sin^2 x \\ B &=& -y\sin x \\ C &=& y^2 \end{cases}$$

$$B^2 - AC = y^2\sin^2 x - y^2\sin^2 x = 0$$

$$\sin^2 x\,\varphi_x\varphi_x - 2\,y\sin x\,\varphi_x\varphi_y + y^2\varphi_y\varphi_y = 0$$

$$\sin^2 x - 2\,y\sin x\,\frac{\varphi_y}{\varphi_x} + y^2\left(\frac{\varphi_y}{\varphi_x}\right)^2 = 0$$

$$\frac{\varphi_y}{\varphi_x} = \frac{y\sin x \pm 0}{y^2} = \frac{\sin x}{y}$$

$$y\,\varphi_y - \sin x\,\varphi_x = 0$$

Karakteristisk ekvation: $-\sin x\,\mathrm{d}y = y\,\mathrm{d}x$:

$$-\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{\sin x}$$

$$-\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{\sin x}$$

$$-\ln y = \ln\left|\tan\frac{x}{2}\right| + \gamma$$

$$\underbrace{-\ln y - \ln\left|\tan\frac{x}{2}\right|}_{\xi = \varphi(x,y)} = \gamma$$

$$\eta = x$$

EXEMPEL:

$$x\,u_{x\,x} + y\,u_{y\,y} = 0, \quad x, y > 0$$

$$\begin{cases} A &= x \\ B &= 0 \\ C &= y \end{cases}$$

$$B^2 - A\,C = 0 - x\,y < 0$$

$$x\,\varphi_x\varphi_x + y\,\varphi_y\,\varphi_y = 0$$

$$x + y\left(\frac{\varphi_y}{\varphi_x}\right)^2 = 0$$

$$\frac{\varphi_y}{\varphi_x} = \mathrm{i}\,\frac{\sqrt{x}}{\sqrt{y}}$$

$$\sqrt{y}\,\varphi_y - \mathrm{i}\sqrt{x}\,\varphi_x = 0$$

$$-\,\mathrm{i}\sqrt{x}\,\mathrm{d}y = \sqrt{y}\,\mathrm{d}x$$

$$\frac{\mathrm{d}x}{\sqrt{x}} = -\,\mathrm{i}\,\frac{\mathrm{d}y}{\sqrt{y}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x}} = -\,\mathrm{i}\int \frac{\mathrm{d}y}{\sqrt{y}}$$

$$2\sqrt{x} = -\,\mathrm{i}\,2\sqrt{y} + \gamma$$

$$2\,\sqrt{x} + \mathrm{i}\,2\sqrt{y} = \gamma$$

$$2\sqrt{x} = \varphi(x, y), \quad 2\sqrt{y} = \psi(x, y)$$