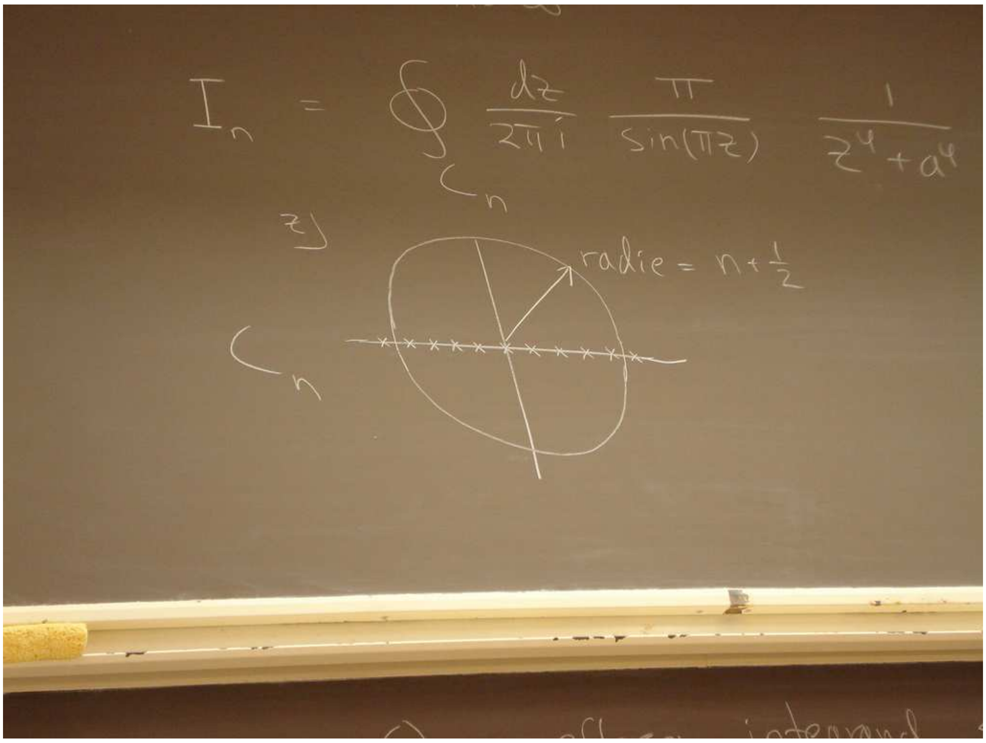
2007–11–15

18.

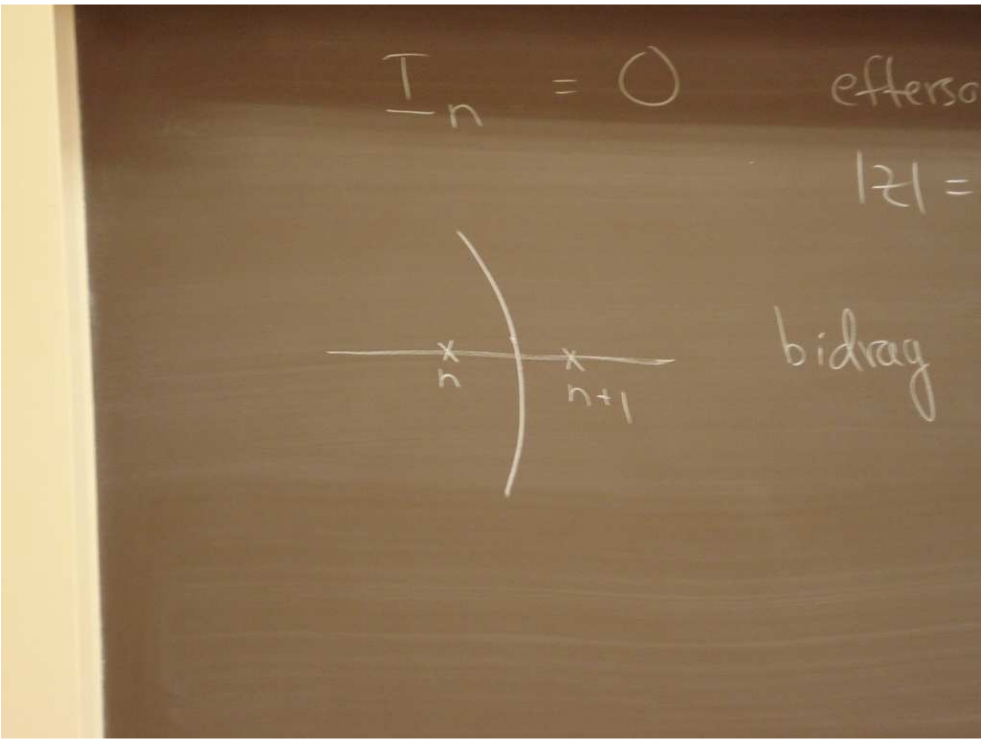
$$S = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^4 + a^4}, \quad a \in \mathbb{R}$$

$$I_n = \oint_{C_n} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\pi}{\sin(\pi z)} \frac{1}{z^4 + a^4}$$



Figur 1.

$I_n \to 0$ då $n \to \infty$ eftersom integrand $< \frac{1}{R}$ för $|z| = R \to \infty$.



Figur 2.

⟨fig2⟩ ger bidrag

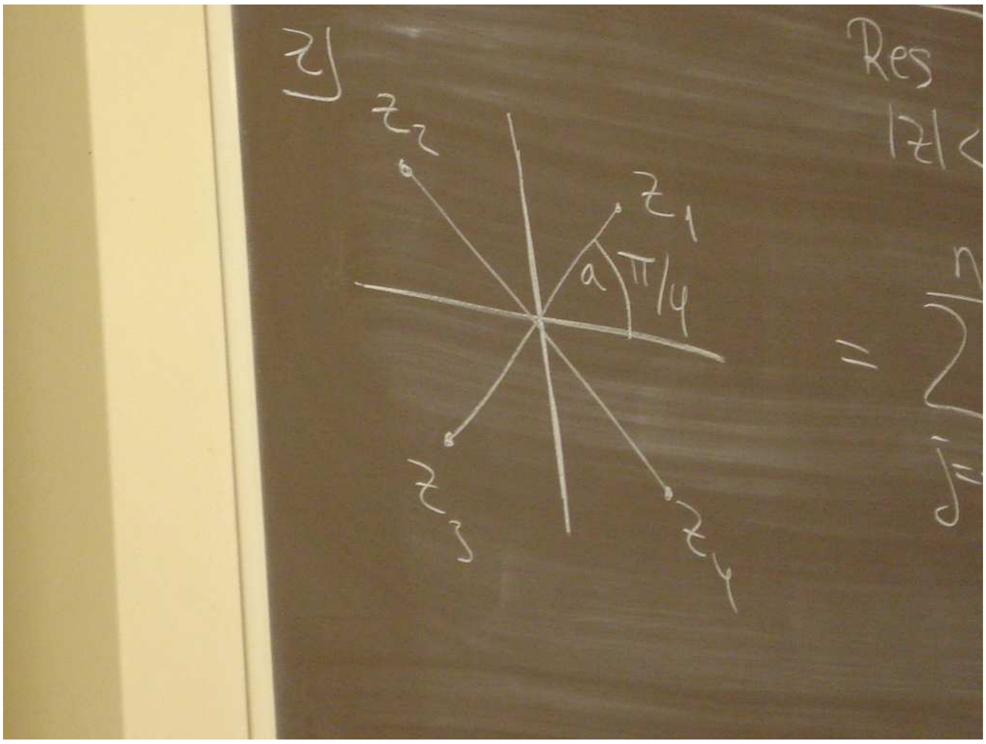
$$\frac{\pi}{\sin\left[\pi\left(n+\frac{1}{2}+\mathrm{i}y\right)\right]}=\frac{\pi}{\sin\left[\pi\left(n+\frac{1}{2}\right)\right]\cos(\mathrm{i}\,y)+\cos\left[\pi\left(n+\frac{1}{2}\right)\right]\sin(\mathrm{i}\,y)}=\frac{\pi}{(-1)^n\cos(\mathrm{i}y)}=$$

$$=\frac{\pi}{(-1)^n\cosh y}$$

$$\Rightarrow |\text{integranden}|\leqslant\frac{\pi}{R^4},\ |z|\to\infty$$

men

$$I_n=\sum_{\mathrm{Res}}\mathrm{Res}\left[\frac{\pi}{\sin(\pi z)}\cdot\frac{1}{z^4+a^4}\right]=\sum_{j=-n}^{n}\frac{(-1)^j}{j^4+a^4}+\sum_{z=z_{1,2,3,4}}\mathrm{Res}\left[\frac{\pi}{\sin(\pi z)}\frac{1}{z^4+a^4}\right]$$



Figur 3.

$$\Rightarrow S=-\sum_{z_j}\mathrm{Res}\left[\frac{\pi}{\sin(\pi z)}\frac{1}{z^4+a^4}\right]$$

använd $z^4+a^4=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$.

$$\begin{cases} z_1 &= \mathrm{e}^{\mathrm{i}\pi/4}\cdot a \\ z_2 &= -z_1^* \\ z_3 &= -z_1 \\ z_4 &= z_1^* \end{cases}$$

$$\Rightarrow S=-\frac{\pi}{\sin(\pi z_1)}\left(\frac{1}{(z_1+z_1^*)(z_1+z_1)(z_1-z_1^*)}\right)-\frac{\pi}{\sin(\pi z_1^*)}\frac{1}{(-z_1^*-z_1)(-z_1^*+z_1)(-z_1^*-z_1^*)}-$$

$$-\frac{\pi}{\sin(-\pi z_1)}\frac{1}{(-z_1-z_1)(-z_1+z_1^*)(-z_1-z_1^*)}-\frac{\pi}{\sin(\pi z_1^*)}\frac{1}{(z_1^*-z_1)(z_1^*+z_1^*)(z_1^*+z_1)}=$$

$$=-\frac{\pi}{\sin(\pi z_1)}\left[\frac{1}{2z_1\left(z_1^2-z_1^{*2}\right)}+\frac{1}{2z_1(z_1^2-z_1^{*2})}\right]-\frac{\pi}{\sin(\pi z_1^*)}\left[\frac{1}{2z_1^*(z_1^{*2}-z_1^2)}+\frac{1}{2z_1^*(z_1^{*2}-z_1^2)}\right]=$$

$$=\frac{\pi}{z_1^2-z_1^{*2}}\left[-\frac{1}{z_1\sin(\pi z_1)}+\frac{1}{z_1^*\sin(\pi z_1^*)}\right]=\frac{\pi}{2\,\mathrm{i}\,a^2}\left[\frac{-z_1^*\sin(\pi z_1^*)}{|z_1|^2\,|\sin(\pi z_1)|^2}+\frac{z_1\sin(\pi z_1)}{|z_1|^2\,|\sin(\pi z_1)|^2}\right]=$$

$$=\frac{\pi}{a^4}\frac{\mathrm{Re}[-\mathrm{i}z_1\sin(\pi z_1)]}{|\sin(\pi z_1)|^2}$$

$$z\sin(\pi z)=(x+\mathrm{i}\,y)\sin(\pi x+\mathrm{i}\pi y)=(x+\mathrm{i}y)\,[\sin(\pi x)\cos(\mathrm{i}\pi y)+\cos(\pi x)\sin(\mathrm{i}\pi y)]=$$

$$=(x+\mathrm{i}y)[\sin(\pi x)\cosh(\pi y)+\mathrm{i}\cos(\pi x)\sinh(\pi y)]=$$

$$=x\sin(\pi x)\cosh(\pi y)-y\cos(\pi x)\sinh(\pi y)+\mathrm{i}\,x\cos(\pi x)\sinh(\pi y)+\mathrm{i}y\sin(\pi x)\cosh(\pi y)$$

$$\Rightarrow S=\frac{\pi}{a^4}\frac{x\cos(\pi x)\sinh(\pi y)+y\sin(\pi x)\cosh(\pi y)}{\sin^2(\pi x)\cosh^2(\pi y)+\cos^2(\pi x)\sinh^2(\pi y)}$$

där $x=\mathrm{Re}\,z_1=a/\sqrt{2}$ och $y=\mathrm{Im}\,z_1=a/\sqrt{2}$.

$$\Rightarrow S=\frac{\pi}{\sqrt{2}\,a^3}\cdot\frac{\cos\left(\frac{\pi a}{\sqrt{2}}\right)\sinh\left(\frac{\pi a}{\sqrt{2}}\right)+\sin\left(\frac{\pi a}{\sqrt{2}}\right)\cosh\left(\frac{\pi a}{\sqrt{2}}\right)}{\cosh^2\left(\frac{\pi a}{\sqrt{2}}\right)-\cos^2\left(\frac{\pi a}{\sqrt{2}}\right)}$$

Uppgift 31

$$I(x)=\int_0^1\mathrm{d}t\cos(x\,t^4)\tan(t)=\mathrm{Re}\int_0^1\mathrm{d}t\tan(t)\,\mathrm{e}^{\mathrm{i}xt^4}$$

Stationär fas:

$$\frac{\partial}{\partial t}\left(x\,t^4\right)=4\,x\,t^3=0\quad\Rightarrow\quad t=0$$

Problem: $\tan(t)=0$ vid $t=0$. $\Rightarrow$ Variabelbyte, t.ex.

(i) $s = t^2 \;\; \Rightarrow \; \mathrm{d}t = \frac{\mathrm{d}s}{2\sqrt{s}} \;\; \Rightarrow$

$$\tan(t)\mathrm{d}t = \frac{\tan\left(\sqrt{s}\right)}{2\sqrt{s}}\mathrm{d}s \quad \neq 0 \;\text{ för } s \to 0$$

(ii) Välj $s = -\ln(\cos t) \;\Rightarrow$

$$\mathrm{d}s = -\frac{-\sin t}{\cos t}\mathrm{d}t = \tan(t)\mathrm{d}t$$

Använd alternativ (ii): $\mathrm{e}^{-s} = \cos t \iff t = \arccos(\mathrm{e}^{-s})$:

$$I(x) = \int_0^{-\ln(\cos 1)} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}x[\arccos(\exp(-s))]^4}$$

Stationär fas:

$$0 = \frac{\partial}{\partial s}\left[\arccos^4\left(\mathrm{e}^{-s}\right)\right] = 4\arccos^3(\mathrm{e}^{-s}) \cdot \frac{-1}{\sqrt{1-\mathrm{e}^{-2s}}} \cdot (-\mathrm{e}^{-s}) =$$

$$= \underbrace{\frac{4\,\mathrm{e}^{-s}}{\sqrt{1-\mathrm{e}^{-2s}}}}_{\neq 0 \forall s} \arccos^3(\mathrm{e}^{-s})$$

$$\arccos(\mathrm{e}^{s}) = 0, \quad \mathrm{e}^{-s} = 1 \quad \Rightarrow s = 0$$

Utveckla runt $s = 0$.

$$\arccos(\mathrm{e}^{-s}) \approx \arccos\left(1 - s + \frac{1}{2}s^2\right)$$

Problem: $\arccos(y)$ har ingen Taylorutveckling runt $y = 1$.

Sätt $u = \arccos\left(1 - s + \frac{1}{2}s^2\right)$. Ta cosinus:

$$\cos u = 1 - s + \frac{1}{2}s^2$$

$$\Rightarrow 1 - \frac{1}{2}u^2 = 1 - s + \frac{1}{2}s^2 \quad \Rightarrow u = \sqrt{2s - s^2}$$

$$\Rightarrow \arccos\left(\mathrm{e}^{-s}\right) = \sqrt{2s}\sqrt{1 - \frac{s^2}{2\,s}} \approx \sqrt{2s}\left(1 - \frac{s}{4}\right)$$

$$\Rightarrow I(x) \approx \mathrm{Re}\int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}x\left[\sqrt{2s}\left(1-\frac{s}{4}\right)\right]^2} = \mathrm{Re}\int_0^{\infty}\mathrm{d}s\, \mathrm{e}^{\mathrm{i}x4s^2} =$$

$$= \mathrm{Re}\left[\frac{1}{2}\sqrt{\frac{\pi}{8x}}\,(1+\mathrm{i})\right] = \frac{1}{4}\sqrt{\frac{\pi}{2\,x}}$$

Uppgift 25 En svår uppgift.

$$y'' = (\ln x)^2\, y$$

Hitta approximativ lösning för $x \to \infty$.

Är $x = \infty$ en ISP? $t = \frac{1}{x} \Rightarrow$

$$y_x = \frac{\mathrm{d}t}{\mathrm{d}x} y_t = -\,t^2\, y_t$$

$$y_{x\,x} = -\,t^2 \partial_t\big(-\,t^2 \partial_t\big)\, y = t^4 y_{t\,t} + 2\,t^3\, y_t$$

$$(\ln x)^2 = \left(\ln \frac{1}{t}\right)^2 = (\ln t)^2$$

$$\Rightarrow y_{t\,t} + \frac{2}{t} y_t - \frac{\ln^2 t}{t^4}\, y = 0$$

$\Rightarrow t = 0$ är en ISP $\Rightarrow x = \infty$ är en irreguljär singulär punkt.

Ansats:

$$y = \mathrm{e}^{S(x)}$$

$$y' = S'\, y$$

$$y'' = \left[S'' + (S')^2\right]\, y$$

Detta ger en ny ekvation för S:

$$S'' + (S')^2 = (\ln x)^2$$

Antag $S'' \ll (S')^2$. Då får vi

$$(S')^2 = (\ln x)^2 \quad \Rightarrow S'(x) = \pm \ln x \quad \Rightarrow \quad S''(x) = \pm \frac{1}{x}$$

och $S'' \ll (S')^2$.

$$S'_{\pm} = -\ln x \quad \Rightarrow \quad S_{\pm}(x) = \pm\, x\,(\ln x - 1)$$

Bättre approximation:

$$S_{\pm} = \pm\, x(\ln x - 1) + C_{\pm}(x)$$

där $C_\pm \ll x \ln x$ då $x \to \infty$.

$$\begin{cases} S'_\pm = \pm \ln x + C'_\pm \\ S''_\pm = \pm \frac{1}{x} + C''_\pm \end{cases}$$

$$\Rightarrow S'' + (S')^2 = \pm \frac{1}{x} + C''_\pm + (\ln x)^2 \pm 2 \ln x \cdot C'_\pm + \left(C'_\pm\right)^2 = (\ln x)^2$$

$$\Rightarrow C''_\pm \pm 2 \ln x \cdot C'_\pm + \left(C'_\pm\right)^2 \pm x = 0$$

Antag att $C''_\pm \ll \left(C'_\pm\right)^2$:

$$\Rightarrow \left(C'_\pm\right)^2 \pm 2 \ln x \cdot C'_\pm \pm \frac{1}{x} = 0$$

$$\Rightarrow C'_{\color{blue}\pm} = {\color{blue}\mp} \ln x \left[1 {\color{red}\pm} \sqrt{1 {\color{blue}\mp} \frac{1}{x\,(\ln x)^2}}\right]$$

De blå ${\color{blue}\pm}$ hör ihop med $C_{\color{blue}\pm}$, medan rött ${\color{red}\pm}$ är de två lösningarna till andragradsekvationen. Där står alltså fyra varianter av C.

Vilket tecken ska det vara framför $\sqrt{}$?

Försök $1 + \sqrt{\cdots} \Rightarrow C'_\pm \approx \pm 2 \ln x \Rightarrow C_\pm \not\ll x \ln x$. Det gick inte. Vi måste välja $1 - \sqrt{\cdots}$.

$$C'_\pm \approx \mp \ln x \left[1 - \left(1 \mp \frac{1}{2} \cdot \frac{1}{x\,(\ln x)^2}\right)\right] = -\frac{1}{2\,x \ln x}$$

$$C''_\pm \approx \frac{1}{2\,x^2 \ln x} + \frac{1}{2\,x^2\,(\ln x)^2} \not\ll \frac{1}{4\,x^2\,(\ln x)^2} = \left(C'_\pm\right)^2$$

Dock: både C'' och $(C')^2$ kan försummas jämfört med $\pm 2 \ln x \cdot C'_\pm$ och $\pm \frac{1}{x}$.

$$\Rightarrow \pm 2 \ln x \cdot C'_\pm \pm \frac{1}{x} = 0$$

$$\Rightarrow C'_\pm = -\frac{1}{2\,x \ln x}$$

$$C_\pm = -\frac{1}{2} \int^x \frac{\mathrm{d}t}{t \ln t} = [\text{sätt } t = \mathrm{e}^y] = -\frac{1}{2} \int^{\ln x} \frac{\mathrm{e}^y \mathrm{d}y}{\mathrm{e}^y\, y} = -\frac{1}{2} \ln(\ln x)$$

En ännu bättre approximation:

$$S_\pm(x) = \pm x\,(\ln x - 1) - \frac{1}{2} \ln(\ln x) + D_\pm(x)$$

där $D_\pm \ll \ln(\ln x)$ för stora x.

$$S'_\pm = \pm \ln x - \frac{1}{2x \ln x} + D'_\pm$$

$$S''_\pm = \pm \frac{1}{x} + \frac{1}{2\,x^2 \ln x} + \frac{1}{2\,x^2\,(\ln x)^2} + D''_\pm$$

$$\Rightarrow \pm \frac{1}{x} + \frac{\ln x + 1}{2\,x^2 (\ln x)^2} + D''_\pm + (\ln x)^2 \mp \frac{1}{x} + \frac{1}{4\,x^2 (\ln x)^2} + 2\,D'_\pm \left[\pm \ln x - \frac{1}{2\,x \ln x} \right] + \left(D'_\pm \right)^2 = (\ln x)^2$$

$$\left(D'_\pm \right)^2 + 2\,D'_\pm \left[\pm \ln x - \frac{1}{2\,x \ln x} \right] + D''_\pm + \frac{\ln x + \frac{3}{2}}{2\,x^2\,(\ln x)^2} = 0$$

Samma sak som för $C'_\pm$: både $D''_\pm$ och $\left(D'_\pm \right)^2$ kan försummas. Dessutom $n x \gg \frac{3}{2}$ och $\ln x \gg 1/(2x \ln x)$:

$$\Rightarrow 2\,D'_\pm(\pm \ln x) + \frac{1}{2\,x^2 \ln x} \approx 0$$

$$\Rightarrow D'_\pm = \mp \frac{1}{4\,x^2\,(\ln x)^2}$$

$$\Rightarrow D_\pm(x) = \mp \frac{1}{4} \int^x \frac{\mathrm{d}t}{t^2\,(\ln t)^2} = \text{konstant} \pm \frac{1}{4} \int_x^\infty \frac{\mathrm{d}t}{t^2\,(\ln t)^2}$$

Sätt

$$I = \int_x^\infty \frac{\mathrm{d}t}{t^2 (\ln t)^2} \int_x^\infty \mathrm{d}t\, \frac{1}{(\ln t)^2} \frac{\partial}{\partial t}\left(-\frac{1}{t} \right) = \int_x^\infty \frac{1}{(\ln t)^2} \cdot \frac{-1}{t} + \int_x^\infty \mathrm{d}t \cdot \frac{1}{t} \cdot \frac{-2}{(\ln t)^3} \cdot \frac{1}{t} =$$

$$= \frac{1}{x\,(\ln x)^2} - 2 \int_x^\infty \frac{\mathrm{d}t}{t^2\,(\ln t)^3}$$

$$\int_x^\infty \frac{\mathrm{d}t}{t^2 (\ln t)^3} \leqslant \frac{1}{\ln x} \int_0^\infty \frac{\mathrm{d}t}{t^2 (\ln t)^2} = \frac{1}{\ln x} \cdot I$$

$$\Rightarrow D_\pm(x) = \text{konstant} - \frac{1}{4} \cdot \frac{1}{x (\ln x)^2} \quad \xrightarrow[x \to \infty]{} \; 0$$

$$y(x) \approx \exp\left(\pm x (\ln x - 1) - \frac{1}{2} \ln(\ln x) \pm \frac{1}{4} \cdot \frac{1}{x\,(\ln x)^2} \right)$$