## 2007–10–31

Avskrivet från Carls anteckningar

**Integrationstekniker:**

– Härleda en differentialekvation och lösa den.

Till exempel,

$$I = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

$$I(\alpha) = \int_0^\infty \mathrm{d}x\,\frac{\mathrm{e}^{-\alpha x}\sin x}{x}$$

$$\Rightarrow I'(\alpha) = -\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\alpha x}\sin x = -\,\mathrm{Im}\left[\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\alpha x+\mathrm{i}x}\right] =$$

$$= -\,\mathrm{Im}\,\frac{1}{\alpha-\mathrm{i}} = -\,\mathrm{Im}\,\frac{\alpha+\mathrm{i}}{\alpha^2+1} = -\frac{1}{\alpha^2+1}$$

$$\Rightarrow I(\alpha) = -\int \frac{\mathrm{d}\alpha}{\alpha^2+1} = -\arctan\alpha + C$$

$$I(\infty) = 0 \quad\Rightarrow\quad C = \frac{\pi}{2}$$

$$\Rightarrow\quad I = I(0) = \frac{\pi}{2}$$

– Tricks:

$$I_0(\alpha) = \int_{-\infty}^\infty \mathrm{e}^{-\alpha x^2}\mathrm{d}x$$

$$I_0^2(\alpha) = \int_{-\infty}^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}y\,\mathrm{e}^{-\alpha(x^2+y^2)} = \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi\,\mathrm{e}^{-\alpha r^2} =$$

$$= 2\pi\,\frac{-1}{2\alpha}\int_0^\infty \mathrm{d}r\left(-2\alpha r\,\mathrm{e}^{-\alpha r^2}\right) = \frac{2\pi}{2\alpha}$$

$$\Rightarrow I_0(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

$$I_2(\alpha) = \int_{-\infty}^\infty \mathrm{d}x\,x^2\,\mathrm{e}^{-\alpha x^2} = -\frac{\partial}{\partial\alpha}I_0(\alpha) = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$I_4(\alpha) = \int_{-\infty}^\infty \mathrm{d}x\,x^4\,\mathrm{e}^{-\alpha x^2} = \frac{\partial}{\partial\alpha}I_2(\alpha) = \frac{1\cdot 3}{(2\alpha)^2}\sqrt{\frac{\pi}{\alpha}}$$

– Symmetri:

exempel:

$$\boldsymbol{I}(\boldsymbol{E}) = \int \mathrm{d}^d p\,\boldsymbol{p}\,(\boldsymbol{p}\cdot\boldsymbol{E})\,\mathrm{e}^{-\alpha p^2}, \quad d = 1, 2, 3, \ldots$$

$\boldsymbol{I}$ är en vektor: måste ha en riktning. Det finns bara en riktning i problemet, $\hat{\boldsymbol{E}}$.

$$\Rightarrow \boldsymbol{I}(\boldsymbol{E}) = (\hat{\boldsymbol{E}}\cdot\boldsymbol{I}(\boldsymbol{E}))\hat{\boldsymbol{E}}$$

$$\hat{\boldsymbol{E}}\cdot\boldsymbol{I}(\boldsymbol{E}) = \int \mathrm{d}^d p\left(\hat{\boldsymbol{E}}\cdot\boldsymbol{p}\right)(\boldsymbol{p}\cdot\boldsymbol{E})\mathrm{e}^{-\alpha p^2} = E\int \mathrm{d}^d p\left(\hat{\boldsymbol{E}}\cdot\boldsymbol{p}\right)^2\mathrm{e}^{-\alpha p^2}$$

Välj koordinatsystem så att $\hat{\boldsymbol{E}} = \hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{E}} = \hat{\boldsymbol{e}}_y$, $\hat{\boldsymbol{E}} = \hat{\boldsymbol{e}}_z$; alla val ger samma resultat.

$\Longrightarrow$ Addera dessa tre svar, och dela med tre (här var $d = 3$).

$$\Rightarrow \boldsymbol{I}(\boldsymbol{E}) = \boldsymbol{E}\,\frac{1}{d}\int \mathrm{d}^d p\, p^2 \mathrm{e}^{-\alpha p^2} = \boldsymbol{E}\,\frac{1}{d}\,S_{d-1}\int_0^\infty \mathrm{d}p\, p^{d-1}\, p^2\, \mathrm{e}^{-\alpha p^2}$$

där $S_{d-1}$ är ytan av en $d$-dimensionell enhetssfär ($S_0 = 2; S_1 = 2\pi; S_2 = 4\pi$ och så vidare).

Till exempel, för $d = 3$,

$$\int_0^\infty \mathrm{d}p\, p^2 p^2\, \mathrm{e}^{-\alpha p^2} = \frac{1}{2}\, I_4(\alpha) = \frac{1}{2}\cdot\frac{3}{4\,\alpha^2}\sqrt{\frac{\pi}{\alpha}}$$

$$\Longrightarrow \boldsymbol{I}(\boldsymbol{E}) = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{3/2}\boldsymbol{E}$$

– extra integraler.

Exempel:

$$I = \int_S \frac{\mathrm{d}^3 k}{(2\pi)^3}\cdot\frac{1}{1 - \frac{1}{3}[\cos k_x + \cos k_y + \cos k_z]}, \qquad S = |k_{x,y,z}| < \pi$$

Använd:

$$\frac{1}{\alpha} = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\alpha z}$$

$$I = \int_0^\infty \mathrm{d}z \int_S \frac{\mathrm{d}^3 k}{(2\pi)^3}\exp\left(-z\left[1 - \frac{1}{3}\cos k_x - \frac{1}{3}\cos k_y - \frac{1}{3}\cos k_z\right]\right) =$$

$$= \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z}\left[\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\frac{z}{3}\cos k}\right]^3$$

Nu kan vi sätta uttrycket inom hakparentesen till $f(z/3)$, där

$$f(\alpha) = \int_{-\pi}^{\pi}\frac{\mathrm{d}x}{2\pi}\,\mathrm{e}^{\alpha\cos x}$$

och utvärdera $f(\alpha)$.

$$f'(\alpha) = \int_{-\pi}^{\pi}\frac{\mathrm{d}x}{2\pi}\cos x\,\mathrm{e}^{\alpha\cos x} = \int_{-\pi}^{\pi}\frac{\mathrm{d}x}{2\pi}\left(\frac{\partial}{\partial x}(\sin x\,\mathrm{e}^{\alpha\cos x}) + \alpha\sin^2 x\,\mathrm{e}^{\alpha\cos x}\right) =$$

$$= \alpha\int_{-\pi}^{\pi}\frac{\mathrm{d}x}{2\pi}\sin^2 x\,\mathrm{e}^{\alpha\cos x}$$

$$f''(\alpha) = \int_{-\pi}^{\pi}\frac{\mathrm{d}x}{2\pi}\cos^2 x\,\mathrm{e}^{\alpha\cos x}$$

$$\Rightarrow f(\alpha) = \frac{1}{\alpha}f'(\alpha) + f''(\alpha)$$

Detta är Bessels ekvation av ordning 0 med imaginära argument.

$$\Rightarrow f(\alpha) = A\,I_0(\alpha) + B\,K_0(\alpha)$$

(Varning! Inte samma $I_0$ som tidigare.)

Men $f(0) = \int_{-\pi}^{\pi} \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{0 \cdot \cos x} = 1$ och

$$\begin{cases} I_0(x) \overset{x \to 0}{\longrightarrow} 1 \\ K_0(x) \underset{x \to 0}{\longrightarrow} \infty \end{cases}$$

$$\Rightarrow I = \int_0^\infty \mathrm{d}z \, \mathrm{e}^{-z} \left[ I_0\left(\frac{z}{3}\right) \right]^3$$

kan inte räknas ut analytiskt (av mig)

$$I \approx 1,51639$$

– Försiktigt med Mathematica!

`N[Integrate[Sqrt[Sin[x]], {x, 0, Pi}]]` = $58.3756 - 1.020356 \cdot 10^{-44}\,\mathrm{i}$

`NIntegrate[Sqrt[Sin[x]], {x, 0, Pi}]` = $2.39628.$