2012–05–28

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This is supposed to be the last session.

Affine Lie algebras: $\hat{\mathfrak{g}}$ (We call an ordinary finite Lie algebra $\mathfrak{g}$.)

Last time we had $\mathfrak{g}_{\text{loop}}$: T^a_m, $\quad T^a \in \mathfrak{g}$.

$$\Rightarrow [T^a_m, T^b_n] = f^{ab}{}_c\, T^c_{m+n}, \quad m \in \mathbb{Z}, n \in \mathbb{Z}.$$

Add a central extension. That's possible since it is allowed by the Jacobi identity.

$K \Rightarrow$

$$\begin{cases} [T^a_m, T^b_n] = f^{ab}{}_c\, T^c_{m+n} + m\delta_{m+n,0}\, \kappa^{ab} K \\ [K, T^a_m] = 0 \end{cases}$$

Cartan–Weyl basis. $\quad H^i \in \mathrm{CSA}(\mathfrak{g})$

$$\left[H^i_m, H^j_n\right] = k\, m\, \delta^{ij}\, \delta_{m+n,0}$$

where k is the eigenvalue of K.

Note: Only $H^i_0 \in \mathrm{CSA}(\hat{\mathfrak{g}})$.

Roots:

$$[H^i_m, E^\alpha_n] = \alpha^i\, E^\alpha_{m+n}$$

$$\left[E^\alpha_m, E^\beta_n\right] = \begin{cases} \varepsilon_{(a,\beta)}\, E^{\alpha+\beta}_{m+n} & \Rightarrow E^{\alpha+\beta}_{m+n} \text{ is a root} \\ \alpha^i H^i_{m+n} + k\, m\, \delta_{m+n,0} & (\alpha = -\beta) \\ 0 & \text{otherwise} \end{cases}$$

$$[K, H^i_m] = [K, E^\alpha_m] = 0$$

Often one imposes hermiticity: $(E^\alpha_m)^\dagger = E^{-\alpha}_{-m}$ etc, and $K^\dagger = K$.

Let's try to find the $\mathrm{CSA}(\hat{\mathfrak{g}})$: H^i_0, K. Enough? (Are these a maximal Cartan subalgebra?)

Roots

$$\begin{aligned} &[H^i_0, E^\alpha_n] = \alpha^i E^\alpha_n \\ &[K, E^\alpha_n] = 0 \end{aligned}$$

$\Rightarrow$ Root of E^α_m: $(\alpha^i, 0)$, which is infinitely degenerated (due to the m index).

But also: $\left[H^i_0, H^j_n\right] = \left[K, H^j_n\right] = 0, n \neq 0$. $\Rightarrow$Root $(H^i_n) = (0, 0)$?

$\Rightarrow H^i_0, K$ is not maximal!

Cure: Add one more generator D:

$$\begin{aligned} &[D, E^\alpha_m] = m\, E^\alpha_m \\ &[D, H^i_m] = m\, H^i_m \\ &[D, K] = 0 \end{aligned}$$

In fact, when combining the affine and Virasoro algebras (Sugawara construction) then $D = -L_0$.

Now things are better: CSA: (H^i_0, K, D).

Roots:

$$\begin{array}{ll} E^{\alpha}_m: & a=(\alpha^i,\ \ 0,\ \ m) \\ H^i_{m\neq 0}: & a=(0,\ \ \ \ 0,\ \ m) \\ & \qquad\quad \uparrow \\ & \qquad \text{always} \\ & \qquad\ \text{zero} \end{array}$$

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What is the Killing form $\hat{\kappa}^{AB}$. $A=(^a_m, k, d)$

Use the Killing form invariance:

$$\hat{\kappa}(x,y): \quad \hat{\kappa}([z,x],y)+\hat{\kappa}(x,[z,y])=0, \quad x,y\in\hat{\mathfrak{g}}$$

1). $z=D, x=T^a_m, y=T^b_m$

$$\Rightarrow \hat{\kappa}(T^a_m, T^b_n)\sim\delta_{m+n,0}$$

2) with $z=T^c_0$ instead

$$\hat{\kappa}(T^a_m, T^b_n)=\hat{\kappa}^{ab}\delta_{m+n,0}$$

$\hat{\kappa}^{AB}\left(\widehat{\text{CSA}}\right)$ is rank $r+2$ with signature $(r+1,1)$.

$$\hat{\kappa}^{AB}=\left(\begin{array}{c|cc} \cdots & & 0 \\ \hline & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)\begin{array}{l} \\ K \\ D \end{array}$$

So: affine KM algebras have Lorentzian signature in the CSA.

Positive roots

1) $(\alpha,0,n), n>0, \forall\alpha\in\mathfrak{g}$

2) $(\alpha,0,0), \alpha>0$ in $\mathfrak{g}$.

Simple roots:

1) α_{simple} in $\mathfrak{g}$: $(\alpha_{\text{simple}},0,0)$.

2) $(-\theta,0,1)$

Example: $A^{(1)}_2$, $\widehat{\text{SU}_k(3)}$. k: eigenvalue of K called level.

$$A_2: \quad A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (\text{SU}(3))$$

$A^{(1)}_2$: simple roots:

$$\begin{cases} a^{(1)}=\left(\alpha^{(1)},0,0\right) \\ a^{(2)}=\left(\alpha^{(2)},0,0\right) \\ a^{(0)}=(-\theta,0,1) \end{cases}$$

Affine Cartan matrix is 3-dimensional (not $2+1+1$ dimensional).

Metric to use

$$\begin{pmatrix} \kappa^{ab} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$V = (v, v_k, v_d)$,

$W = (w, w_k, w_d)$

$$\Rightarrow V \cdot W = v \cdot w + v_k v_d + v_d w_k$$

$$a^{(1)} \cdot a^{(1)} = a^{(2)} \cdot a^{(2)} = 2$$

$$a^{(0)} \cdot a^{(0)} = (-\theta) \cdot (-\theta) = 2$$

$$a^{(1)} \cdot a^{(2)} = -1$$

$$a^{(0)} \cdot a^{(1)} = (-\theta) \cdot \alpha^{(1)} = -\alpha^{(1)} \cdot \alpha^{(1)} - \alpha^{(1)} \cdot \alpha^{(2)} = -2 - (-1) = -1.$$

$$A_2^{(1)}: \quad A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow \det A = 0$$

Check!

Problem $\mathfrak{sp}(2;\mathbb{R})$ is isomorphic as a real Lie algebra to another one. Which one? Prove it.

$$\mathfrak{sp}(2, \mathbb{R}) \sim C_2$$

Could be isomorphic to B_2. Same Dynkin diagram. $B_2 \approx \mathrm{SO}(5)$, dim $\frac{5 \times 4}{2} = 10$.

$\mathfrak{sp}(2, \mathbb{R})$: 4×4 real matrices M that leave invariant the symplectic form J

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}: \quad MJ + JM^T = 0$$

$$MJ = \begin{pmatrix} -B & A \\ -D & C \end{pmatrix}$$

$$JM^T = \begin{pmatrix} B^T & D^T \\ -A^T & -C^T \end{pmatrix}$$

$$\Rightarrow B = B^T, \quad C = C^T, \quad D = -A^T, \quad A = -D^T$$

A is independent 2×2, dim $= 4$. B: 3, $C = 3$. Sum $= 10$.

So C_2 is isomorphic to D_2 as complex.

Now, real case: C_2: $\mathfrak{sp}(2, \mathbb{R})$.

B_2 can be $\mathrm{SO}(5), \mathrm{SO}(4,1), \mathrm{SO}(3,2)$. $\mathrm{SO}(5)$ is compact, but $\mathfrak{sp}(2, \mathbb{R})$ is not. $\mathrm{SO}(4,1)$ and $\mathrm{SO}(3,2)$ are left.

Find an explicit representation of these two Lie algebras that is "the same" as the fundamental representation of $\mathfrak{sp}(2, \mathbb{R})$. It must be a spinor representation.

From spinor representation in any even dimension d one can easily construct spinor representation in $d+1$.

SO(4): $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. $i, j = 1, 2, 3, 4$.

SO(5): $\{\gamma^I, \gamma^J\} = 2\,\delta^{IJ}$, $I, J = 1, 2, 3, 4, 5$. Generators $\gamma^{[I}\gamma^{J]} \equiv \gamma^{IJ}$.

SO(4): $\gamma^5 = \gamma^1\gamma^2\,\gamma^3\,\gamma^4 \Rightarrow \gamma^I = (\gamma^i, \gamma^5)$. $(\gamma^5)^2 = 1$.

SO(1,3): Dirac spinors: 4-component, complex. Can divide into Weyl and anti-Weyl. You can also divide into Majorana spinors: 4-component, real.

Is it consistent to have Weyl and Majorana at the same time? Not possible for SO(1,3).

Use the Majorana representation for SO(1,3) to construct a real spinor representation in 5 dimensions. γ^μ: Majorana, real. $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow (\gamma^5)^2 = -1$. $\Rightarrow$SO(1,3) Majorana $\Rightarrow$ SO(2,3).

σ^1, σ^2 real, σ^2 imaginary, $\mathrm{i}\,\sigma^2 = \varepsilon$ real.

$$\begin{cases} \sigma^\mu = (\mathbb{1}, \sigma^i) \\ \bar{\sigma}^\mu = (-\mathbb{1}, \sigma^i) \end{cases}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = (\varepsilon \otimes \mathbb{1}, \sigma^1 \otimes \sigma^i)$$

Unitarity

In physics the symmetry is often (in quantum mechancis) realized in a unitary fasion. If the Lie algebra in question is compact (as QM, not QFT), all finite dimensional finite dimensional representations are unitarizable. But in QFT one often has non-compact groups (like the Lorentz group) then all unitary representations are infinite-dimensional.

Consider SU(2) and SU(1,1) $\approx$ SO(2,1)

Use a harmonic oscillator realization:

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad i = 1, 2$$

SU(2): we have $J_\pm$ and J_3.

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3$$

Put $J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)$, $J_+ = a_1^\dagger a_2$, $J_- = a_2^\dagger a_1$.

Note: $J_3^\dagger = J_3$, $J_\pm^\dagger = J_\mp$. Compact: all three generators. By this we mean $\mathrm{i}J_3$.

SU(1,1) $K_3 = J_3$. $K_+ = \mathrm{i}\,a_1^\dagger a_2$. $K_- = \mathrm{i}\,a_2^\dagger a_1$.

SU(2).

$$a_i^\dagger|n_i\rangle = \sqrt{n_i + 1}\,|n_i + 1\rangle$$

$$|n_i\rangle \sim \left(a_i^\dagger\right)^n|0\rangle, \quad a_i|0\rangle = 0$$

$$a_i|n_i\rangle = \sqrt{n_i}\,|n_i - 1\rangle$$

$$|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$$

$$\Rightarrow J_3|n_1, n_2\rangle = \frac{1}{2}(n_1 - n_2)|n_1, n_2\rangle$$

Note:

$$N \equiv \frac{1}{2}(n_1 + n_2) \Rightarrow N\,|n_1, n_2\rangle = \frac{1}{2}(n_1 + n_2)|n_1, n_2\rangle$$

Single-valuedness: $e^{i\phi J_3} = e^{i(\phi + 4\pi) J_3}$.

$$n_1 - n_2 \in \mathbb{Z}$$

$$J_3 \in \frac{1}{2}\mathbb{Z}$$

The eigenvalue of J_3 is called m.

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Unitarity

$$\langle n_1 + 1, n_2 + 1|J_+|n_1, n_2\rangle = \sqrt{(n_1 + 1)n_2}$$

Compare to J_-

$$\langle n_1, n_2|J_-|n_1 + 1, n_2 - 1\rangle = \sqrt{(n_1 + 1)n_2}$$

Hermitian conjugate on the J_+ expression gives you J_- using $(J_+)^\dagger = J_-$.

Generalization Let n_1, n_2 be also both negative.

$$\langle n_1 + 1, n_2 - 1|K_+|n_1, n_2\rangle = i\sqrt{(n + 1)\,n_2}$$

$(K_+)^\dagger = K_- \Rightarrow \sqrt{-(n_1 + 1)\,n_2}$ must be real. One n is negative, and one is positive.

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Discrete series $\mathcal{D}_+^j, \mathcal{D}_-^j$. Principle representation $\mathcal{D}^p$, $j + \frac{1}{2} = p$: $-\frac{1}{2} \leqslant p \leqslant \frac{1}{2}$. Doesn't stop in either direction.

Complementary: $j + \frac{1}{2} = i\,\beta$.