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(Next week: no lecture. After that two more lectures. And one more home problem. Or two.)

Chapter 8: Real Lie algebras (sometimes called *real forms*)

In the begining of the course, we talked about various simple Lie algebras, such as

$$\mathfrak{sl}(2,\mathbb{R}) = \{M\colon \operatorname{tr} M = 0,\ 2\times 2 \text{ real matrices}\}.$$

One version (i):

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Another version (ii):

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\kappa^{ab}_{(\mathrm{i})} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \kappa^{ab}_{(\mathrm{ii})} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Related by diagonalization and renormalizing the generators.

Note: From (ii) we see that $\mathfrak{sl}(2,\mathbb{R})$ has one compact and two non-compact elements.

$$\mathfrak{sp}(1,\mathbb{R}) \approx \mathfrak{sl}(2,\mathbb{R}) \approx \mathfrak{su}(1,1)$$

Now: SU(2): Then we have 2×2 complex matrices, with $\det = 1$. Lie algebra $\mathfrak{su}(2)$: antihermitian with zero trace. Easy enough to tabulate:

$$\mathfrak{su}(2) = \left\{ \frac{\mathrm{i}}{2}\sigma^i \right\}$$

The factor 2 is conventional, to get the normalization of the Lie algebra in a nice form.

$$\kappa^{ab} \propto \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We see that all three generators are compact. How to see compact:

$$\mathrm{e}^{\alpha\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)} \sim \mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha \text{ is compact.}$$

Comes out as a minus sign in the Killing form.

This is the same as for $\mathfrak{so}(3)$.

Example of isomorphisms.

$$\mathfrak{so}(1,2) = \{T^a\} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$$\kappa^{ab} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & -2 \end{pmatrix}$$

$$[T^1, T^2] = T^3, \quad [T^2, T^3] = -T^1, \quad [T^3, T^1] = -T^2$$

$\mathfrak{su}(1,1)$:

$$\begin{cases} T^1 = \mathrm{i}\sigma_3 \\ T^2 = \sigma_1 \\ T^3 = -\sigma_2 \end{cases} \Rightarrow \begin{cases} [T^1, T^2] = 2\,T^3 \\ [T^2, T^3] = -2\,T^1 \\ [T^3, T^1] = -2T^2 \end{cases}$$

$\mathfrak{sp}(1,\mathbb{R})$:

$$\begin{cases} [S^1, S^2] = 2\,S^3 \\ [S^2, S^3] = -2\,S^1 \\ [S^3, S^1] = 2\,S^2 \end{cases}$$

You can't stare at the algebra here and say this has to be $\mathfrak{su}(1,1)$. Go to Dynkin and then to the Killing form.

$$\begin{cases} S^2 = T^2 \\ S^1 = T^3 \\ S^3 = T^1 \end{cases}$$

$\mathfrak{sl}(2,\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = H$$

It's really $\mathfrak{sl}(2,\mathbb{R})$ that we have here, since there is no i anywhere. With coefficients in $\mathbb{C}$, this becomes the complex $A_1 \approx \mathfrak{sl}(2,\mathbb{C})$ algebra.

Real forms:

i) Coefficients $\mathbb{C} \to \mathbb{R}$. Restrict them to be real.

$$\Rightarrow \mathfrak{sl}(2,\mathbb{R}): \quad \kappa^{ab} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

This is called the split form (or normal). This is the maximal non-compact real form of A_1.

ii) maximal comapct case: here $\mathfrak{su}(2)$. We have to introduce an i somewhere.

$H, E_+, E_- \to E_+ - E_-$ is compact. Here $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which is obviously compact.

$H, E_+ + E_-$ non-compact. Multiply by i, then these become compact: $\mathrm{i}H, \mathrm{i}(E_+ + E_-)$.

This is then $\mathfrak{su}(2)$.

$\mathfrak{sl}(2,\mathbb{C})$: this can be viewed as a six-dimensional real algebra:

$$X \in \mathfrak{sl}(2,\mathbb{C}): \quad X = \sum_{i=1}^{3} \lambda_i T^i \text{ with } \lambda_i \in \mathbb{C} \text{ and } T^i \text{ compact } (\mathfrak{su}(2))$$

$$\lambda_i = a_i + \mathrm{i}\,b_i$$

$$\Rightarrow X = \sum_i a_i T^i + \sum_i b_i \underbrace{(\mathrm{i}\,T^i)}_{\equiv U^i}$$

$(\mathrm{i}T^i)$ are new generators in the real algebra. We call them U^i.

$$\begin{cases} [T^i, T^j] = \varepsilon^{ijk} T^k \\ [T^i, U^j] = \varepsilon^{ijk} U^k \\ [U^i, U^j] = -\mathrm{e}^{ijk} T^k \end{cases}$$

This is the Lorentz algebra, $\mathfrak{so}(1,3)$.

$\Rightarrow \mathfrak{so}(1,3) \approx \mathfrak{sl}(2,\mathbb{C})$.

Note: Consider x^μ (coordinates in special relativity), and form $X = x^\mu \sigma_\mu, \sigma_\mu = \{\mathbb{1}, \sigma_i\}$.

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}\, x^2 \\ x^1 + \mathrm{i}\, x^2 & x^0 - x^3 \end{pmatrix}$$

$$\det X = -\eta_{\mu\nu} X^\mu X^\nu$$

Note: Dirac γ matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \bar{\sigma}_\mu = (-\mathbb{1}, \sigma_i)$$

$$\Rightarrow \{\gamma_\mu, \gamma_\nu\} = 2\, \eta_{\mu\nu}$$

$$\gamma^{\mu\nu} = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu \end{pmatrix}$$

$$(\gamma^\mu)_A{}^B\!: \quad \rightarrow \quad \underbrace{\Lambda^\mu{}_\nu}_{\mathrm{SO}(1,3)} \ \omega\, \gamma^\nu \omega^{-1} = \gamma^\mu$$

Lie algebra: $\varepsilon_{\rho\sigma} (\delta^{\rho\sigma})^\mu{}_\nu$, $\omega = \frac{1}{4} \varepsilon_{\rho\sigma} \gamma^{\rho\sigma}$. $(\gamma^\mu)_A{}^B$ are invariant matrices (Cl.–Gordan coefficients). spin $\times$ spin $=$ vector.

Killing form $\kappa(T^a, T^b) = \kappa^{ab} = \mathrm{tr}(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}) \sim \mathrm{tr}\, f^a f^b$.

- $\kappa^{ab} = \kappa^{ba}$.
- $\kappa(T^a, [T^b, T^c]) = \kappa([T^a, T^b], T^c) \Rightarrow$ Theorem κ^{ab} is unique up to normalization.

$\mathrm{ad}_{T^b} \kappa(T^a, T^c) = 0$.

Real forms

	complex	compact	split
A_{n-1}	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	$\mathfrak{sl}(n,\mathbb{R})$

$\mathfrak{sl}(n,\mathbb{R})/K \approx$ hyperbolic space. K: maximal compact subgroup $(\mathrm{SO}(n))$.

	complex	compact	split	
A_{n-1}	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	$\mathfrak{sl}(n,\mathbb{R})$	
D_n		$\mathfrak{so}(2\,n), \mathrm{so}(p,q)$	$\mathfrak{so}(n,n)$	where $p+q = 2\,n$

$$E_8, \quad \underset{\substack{\text{compact} \\ \dim = 248}}{E_{(8,-248)}}, \quad E_{(8,-24)}, \quad E_{(8,8)}$$

dim = 248. There is a subgroup SO(16) that had dimension 120.

$$\begin{array}{cccccl} 8, & 120 & + & 120 & \Rightarrow & E_+ - E_-:\ 120\ \ (\mathrm{SO}(16)) \\ H^i & E_+ & & E_- & & \end{array}$$

$\Rightarrow i$ on $H, E_+ + E_-$: 128 non-comact.

Chapter 9: Loop algebras

This has many aspects, that are important and come up in different situations. This is related to, as we will see, affine algebras ($\det A = 0$, Kac–Moody). We want to, in a logical way, construct them.

Appear in string theory, leads to CFT in two dimensions. Also in phase transitions (virasoro).

Recall: All finite-dimensional Lie algebras can be related to finite-dimensional matrices.

How do we construct infinite-dimensional Lie algebras, like the affine?

First step: Construct loop algebras.

Consider a finite-dimensional Lie algebra — call it $\bar{\mathfrak{g}}$ — with basis $\bar{\mathcal{B}} = \{T^a, a = 1, 2, \ldots, \dim \bar{\mathfrak{g}}\}$.

Now we generalize the $\mathfrak{sl}(2, \mathbb{C})$ construction as a real Lie algebra: generators T^a, $a = 1, 2, 3$.

$$X = z_a T^a = (a_a + \mathrm{i}\, b_a)\, T^a = a_a T^a + b_a\, (\mathrm{i}\, T^a) \to \text{a six-dimensional Lie algebra.}$$

Generalization: $z_a \to f_a(z)$ where $z \in S^1$, $z = \mathrm{e}^{\mathrm{i}\theta} \Rightarrow$ expand $f_a(z)$ in terms of mode functions $\mathrm{e}^{in\theta}$, i.e. $f_a(z)\, T^a = \sum_{n\in\mathbb{Z}} \alpha_a{}^n z^n\, T^a = \sum_{n\in\mathbb{Z}} \alpha_a^n \tilde{T}_n^a$ where $\tilde{T}_n^a \equiv T^a \otimes z^n$.

Note: This way we have maps $S^1 \to$ group.

$$g(z) = \mathrm{e}^{\sum f_a(z) T^a}$$

The loop algebra

$$\left[\tilde{T}_m^a, \tilde{T}_n^b\right] = [T^a \otimes z^m, T^b \otimes z^n] = [T^a, T^b] \otimes z^{m+n} = f^{ab}{}_c\, T^c \otimes z^{m+n} = f^{ab}{}_c\, \tilde{T}_{m+n}^c \equiv f^{ambn}{}_{cp}\, \tilde{T}^c{}_p$$

$$\Rightarrow f^{ambn}{}_{cp} = f^{ab}{}_c\, \delta_{m+n,p}$$

This is called $\bar{\mathfrak{g}}_{\text{loop}}$.

Note: The zero-mode algebra is $\bar{\mathfrak{g}}$.

In order to use this algebra in physics we need unitary representations. But (as for the Witt algebra) it has no such representations. We need to generalize it with a central term.

The way to get an affine algebra out of the loop algebra is to add some generators keeping the Jacobi indentity.

The affine algebra has a central term (central extension) which the loop algebra does not have. To see this property in the affine case, recall that the affine Cartan matrix has $\det A = 0$. Null vector: $A\, \vec{\boldsymbol{a}} = 0$ or $a_i\, A^{ji} = 0$. So form

$$K = \sum_{i=0}^{r} a_i\, H^i$$

$$\Rightarrow [K, H^i] = 0$$

$$[K, E_\pm^i] = \sum_i a_i \underbrace{\left[H^i, E_\pm^j\right]}_{= \pm A^{ji} E_\pm^j} = 0$$

i.e. K commutes with the whole algebra, including itself. It is a central element. Canonical central element.

Try to get the loop algebra to pick up this property: add a central term (drop tilde)

$$[T_m^a, T_n^b] = f^{ab}{}_c\, T_{m+n}^c + (f^{ab}{}_i)_{mn} K^i$$

where i runs over the set of K's, if there are many.

Is this possible without violating the Jacobi? Yes. But some K^i are (a) impossible, and some "trivial" (b).

(a). $(f^{ab}{}_i)_{00} = 0$ since $\bar{\mathfrak{g}}$ has no such central term.

(b). Redefining $T^a{}_n \to T^a{}_n - (f^{ab}{}_i)_{0n} K^i$ we can put $(f^{ab}{}_i)_{0n} = 0$.

The last case: $(f^{ab}{}_i)_{mn} K^i$. These are not trivial in the above sense. But if nonzero, they must be tensors in $\bar{\mathfrak{g}}$. We must be able to write this object in terms of tensors in $\bar{\mathfrak{g}}$.

$$(f^{ab}{}_i)_{mn} = \bar{\kappa}^{ab} (f_i)_{mn}$$

$(f_i)_{mn}$: mn: antisymmetric. $\delta_{m,n}$, $m\, \delta_{m+n,0}$.

$$(f^{ab}{}_i)_{mn} = \bar{\kappa}^{ab} (f_i)_{mn} = \bar{\kappa}^{ab}\, m\, \delta_{m+n,0}$$

$\hat{\mathfrak{g}}$:

$$\begin{cases} [T_m^a, T_n^b] = f^{ab}{}_c\, T^c{}_{m+n} + m\, \delta_{m+n,0} \bar{\kappa}^{ab} K \\ [K, T_m^a] = 0 \end{cases}$$

To understand this better we need one more generator D: $[D, T_m^a] = m\, T_m^a$, $[D, K] = 0$.

$$\mathfrak{g} \equiv \{T_m^a, K, D\} = \{\hat{\mathfrak{g}}, D\}$$

with T_m^a from $\bar{\mathfrak{g}}_{\text{loop}}$.

The Killing form

$$\kappa = \left(\begin{array}{c|c} \bar{\kappa}^{ab}\, \delta_{m+n,0} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right)$$

Matrix ordered as $(\mathfrak{g}_{\text{loop}}, K, D)$.

$\kappa(D, D) = 0$ by assumption.