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Conjugacy classes: If you have a lattice Λ, and you take its dual Λ^*, (e.g. root lattice Λ_R and the weight lattice Λ_W), then $\Lambda_R \subset \Lambda_W$: $\Lambda_W/\Lambda_R = I$ (table v). That will give you the number of conjugacy classes.

D_n. Λ_v. $(1,0,\ldots,0)+\Lambda_R$. Λ_v strictly speaking not a lattice, because it doesn't contain the origin.

$$\Lambda_W = \Lambda_R \cup \Lambda_V \cup \Lambda_S \cup \Lambda_C$$

(root, vector, spinor, cospinor).

D_n: the index is always $I=4$.

If you have D_n,

A_n: each node in the Dynkin diagram is a conjugacy class, but that's not the case in D_n.

The four conjugacy classes form a discrete group, with four elements. Either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Dynkin

A_n	$\circ - \circ \ldots \circ - \circ$	$\mathfrak{sl}, \mathfrak{su}$
B_n	$\circ - \circ \ldots \circ = < = \circ$	$\mathfrak{so}(2n-1)$
C_n	...	$\mathfrak{sp}$
D_n	...	$\mathfrak{so}(2n)$
E_6	...	
E_7	...	
E_8	...	
G_2	$\circ \equiv > \equiv \circ$	
F_4	$\circ - \circ = > = \circ - \circ$	

• A_n, D_n, $E_{6,7,8}$ ("ADE classification") are simply laced (single line diagrams $\leftrightarrow$ all roots of the same length)

• Singularities ⟨fig⟩ $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1 \Rightarrow$ ADE.

• Modular invariants in CFT.

• CFT in 6 dimensions.

• Discrete subgroups of SU(2).

Representation theory: ⟨fig⟩.

The highest root Θ ($\equiv$ highest weight of the adjoint).

• Deleting a node $\Rightarrow$ subalgebra.

$$A_2 \oplus D_5 \subset E_8$$

Careful with U(1) factors.

• There is a special sequence relevant in elementary particle model building and Kaluza–Klein supergravity: Remove the node to the far left one by one:

$$\begin{array}{ccccccccc} & & \mathfrak{so}(10) & & \mathfrak{su}(5) & & \mathfrak{su}(2)\times\mathfrak{su}(3) & & \mathfrak{su}(2)\times\mathfrak{su}(2) \\ E_8 \to E_7 \to E_6 & \to & D_5 & \to & A_4 & \to & A_1 \oplus A_2 & \to & A_1 \oplus A_1 \\ & & (E_5) & & (E_4) & & (E_3) & & (E_2) \end{array}$$

Some Dynkin are more symmetric than others ⟨fig⟩.

• Low rank isomorphisms (complex case). ⟨fig⟩

Classification of self-dual lattices: (consistency of any string theory $\equiv$ self-duality).

Euclidean lattice: exist only in $8n$ dimensions.

$d=8$: E_8.

$d=16$: $E_8\times E_8$ and $\mathrm{SO}(32)/\mathbb{Z}_2$.

$d=24$: 24 cases (Niemeier lattices). 23 of them come from Lie algebras. The last one has no vectors of $|\,|^2=2$! The shortest ones: $|\,|^2=4$. Leech lattices $\rightarrow$ Monster!

$d>24$: zillings of cases.

Levi's theorem (classification of general Lie-algebras

Real ones! Example: $\mathfrak{sl}(3,\mathbb{R})$, simple, ∘–∘, maximal compact subalgebra, 3×3 matrices with zero trace.

$$\text{Cartan:}\quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Step operators

$$\begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix}:\quad E_+$$

$$\rightarrow(\quad)^T: E_-$$

$$E_+^{(1)}-E_-^{(1)}=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Antisymmetric. Compact. $g=\mathrm{e}^{\alpha E^{(1)}}\sim\cos(\alpha)+E^{(1)}\sin\alpha$. Compact.

Maximal compact algebra is $\mathfrak{so}(3)$.

Subexample: $\mathfrak{sl}(3,\mathbb{R})/\mathfrak{so}(3)\approx$ hyperbolic space.

Example. Consider the Lie algebra generated by

$$\begin{pmatrix} 0 & a & \hbar \\ 0 & N & a^\dagger \\ 0 & 0 & 0 \end{pmatrix}$$

with $N, a, a^\dagger, \hbar$ real parameters here!

$$=a\,X_a+a^\dagger\,X_{a^\dagger}+N\,X_N+\hbar\,X_\hbar$$

where

$$X_a=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

etc, with the lie algebra:

$$[X_N, X_{a^\dagger}]=X_{a^\dagger}$$

$$[X_N, X_a]=-X_a$$

$$[X_a, X_{a^\dagger}]=X_\hbar$$

$$[X_N, X_\hbar]=0$$

(The other ones are zero.)

Given the algebra with structure constants ${f^{ij}}_k$, $i = N, a, a^\dagger, \hbar$.

Example

$$ {f^{Ni}}_j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{etc} $$

The X's are a matrix realization, while the operator realization of the same algebra is $X_N \to \hat{a}^\dagger \hat{a}$, $X_a \to \hat{a}$, $X_{a^\dagger} \to \hat{a}^\dagger$, $X_\hbar \to 1$ with $[\hat{a}, \hat{a}^\dagger] = \hat{1}$.

Then $\kappa^{ij} = \mathrm{Tr}(f^i f^j)$ (where $f^i \equiv {(f^i)^j}_k$)

$$ = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} $$

Example:

$$ \mathrm{SU}(2) \times \mathrm{U}(1) \Rightarrow \kappa^{ij} = \left(\begin{array}{c|c} \mathfrak{su}(2) & 0 \\ 3 \times 3 & \\ \hline 0 & 0 \end{array}\right) $$

Theorem: Consider a general Killing form. The zero-eigenvalue subspace (rank $<$ dim) is the maxima *nilpotent* part. Remove this subspace (here a, $a^\dagger$, h) and compute κ^{ij} again. The zero eigenvalue subspace this time is the abelian part. The rest is semi-simple.

Levi's theorem

$$ \mathfrak{g} = \underbrace{V_-^{\text{compact}} \oplus V_+^{\substack{\text{non-} \\ \text{compact}}}}_{\substack{\text{semi-simple} \\ = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \\ \text{(simple)}}} \oplus \underbrace{V_0^{\text{abelian}} \oplus V_0^{\text{nilpotent}}}_{\substack{\text{maximal solvable} \\ (\equiv \text{ radical})}} $$

Chapter 8: Real Lie algebras (or real forms)

First

$$ \mathfrak{sl}(2, \mathbb{R}) = \{ M : 2 \times 2 \text{ real matrices, } \mathrm{tr}\, M = 0 \} $$

$$ \text{(i)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \quad \Rightarrow \kappa^{ab} = \mathrm{Tr}(\ldots) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} $$

$$ \text{(ii)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \quad \Rightarrow \kappa^{ab} = \mathrm{Tr}(\ldots) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} $$

$\mathfrak{sp}(1, \mathbb{R})$

This is 2×2 matrices, leaving invariant the symplectic matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

g in the group $\mathrm{Sp}(1, \mathbb{R})$: $g J g^T = J$.

$g = e^M \approx 1 + M \Rightarrow (1 + M)\, J\, (1 + M^T) = J \Rightarrow$

$$M J + J M^T = 0$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{cases} a = -d \\ b = b \\ c = c \end{cases} \Rightarrow M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$\approx \mathfrak{sl}(2, \mathbb{R})$$

$\mathfrak{su}(1,1)$. The group g: 2×2 complex, leave the metric $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ invariant.

If we have $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$: $z \to g\, z$, $z^\dagger \to z^\dagger g^\dagger$ then

$$g^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det g = 1$$

Then

$$g = e^A = 1 + A + \cdots \Rightarrow \operatorname{tr} A = 0$$

$$A^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = 0.$$

Set

$$A^\dagger = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$A = \begin{pmatrix} \mathrm{i}\, a & \beta \\ \beta^* & -\mathrm{i} a \end{pmatrix} \text{ with } a \in \mathbb{R}.$$

Basis

$$= \left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \right\}$$

$$\Rightarrow \kappa^{ab} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example: $\mathfrak{su}(2)$:

$$\kappa^{ab} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

So negative diagonal Killing elements $\Leftrightarrow$ compact generators.