2012–04–25

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Recall: Generalised Cartan matrix A. Properties:

- $A^{ii} = 2$ (diagonal elements equal 2)
- $i \neq j$: $A^{ij} = 0 \Leftrightarrow A^{ji} = 0$.
- $i \neq j$: $A^{ij} \in \mathbb{Z}_{(\leqslant 0)}$.
- a) $\det A > 0 \Rightarrow$ finite Lie algebras. Completely classified.

  b) $\det A = 0$

  c) $\det A < 0$

  b and c: Kac–Moody: affine (classified), hyperbolic (classified), Lorentzian etc (not fully classified)

Cartan class: Start from rank 2: $A_1 \otimes A_1$, A_2, B_2, G_2 and using a step-by-step procedure one can get all rank 3, 4, etc.

$$\det \left(\begin{array}{c|cc} 2 & \alpha & \beta \\ \hline a & & \\ b & & A_{(2)} \end{array} \right) > 0, \quad \alpha, \beta, a, b$$

We also saw that the angle between two simple roots can only take the values

$$\begin{array}{ll} A^{ij} = 0 & \theta = \pi/2 \\ A^{ij} = -1 & \theta = 2\pi/3 \\ A^{ij} = -2 & \theta = 3\pi/4 \\ A^{ij} = -3 & \theta = 5\pi/6 \end{array}$$

We will go, today, to Dynkin diagrams.

Recall also, in the case of (b) above: Affine Lie algebras: A^{ij}: $(r+1) \times (r+1)$, with $i = 0, 1, 2, \ldots, r$.

- $\det A^{ij} = 0$
- $\mathrm{rank}(A) = 0$, $\det A_{(i)} > 0$ where $A_{(i)}$ denotes the matrix A with row i and column i deleted.

The second property means that any affine Lie algebra can be obtained from a simple finite-dimensional one (class (a) above), by adding to it an extra simple root called the *extended root*.

Note: The class (a): A^{ij} is Euclidean — if you diagonalize it you get positive integers. Class (b) — what can that be? Why don't you go home and diagonalize A_2, G_2, etc. In class (b), if you diagonalize it, since $\det A = 0$ one eigenvalue has to be zero.

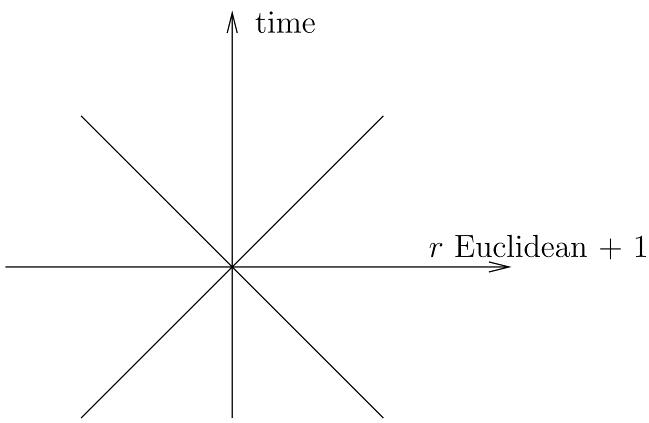


Figure 1. Take rank r. Extend this space with two dimensions — one which is space-like and one which is light-like — so it becomes Lorentzian. This algebra is hyperbolic. It has a Lorentzian Cartan–Killing form, or Cartan matrix. *(Continued in figure 2.)*

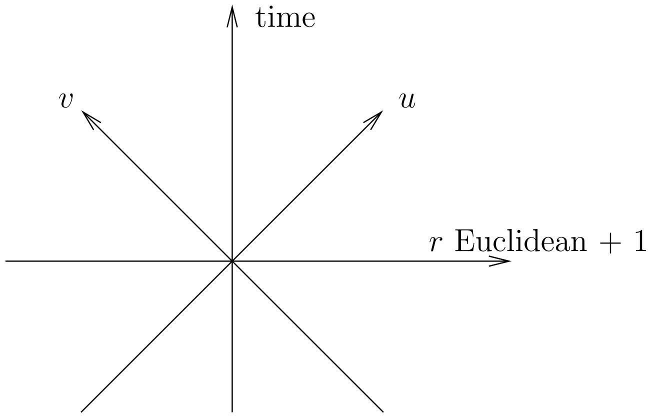
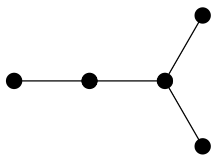


Figure 2. Now you can just take — we have done this, you do this all the time: If you have $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$ you go over to $\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v$: light-cone coordinates. So u goes up right and v goes up left in the figure. But suppose you forget about v and just add u, then the $-\mathrm{d}u\,\mathrm{d}v$ term goes away; well we have $\mathrm{d}x_i^2$ here (that's the rank r terms): $\mathrm{d}x^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}x_i^2 = -\mathrm{d}u\mathrm{d}v + \mathrm{d}x_i^2$. If you forget about v, then the $-\mathrm{d}u\,\mathrm{d}v$ sort of disappears. $\mathrm{d}s^2 = \mathrm{d}x_i^2$. There is one coordinate (u) that just drops out; so this is degenerate, this metric. That's the affine case. So this extended root is sort of light-like. Then you can add another light-like (v), to get to the over-extended case; but then you can diagonalize it, and then it becomes Lorentzian. So that's the hyperbolic.

7.3. Dynkin diagrams

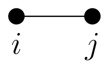
To find all Cartan matrices is easy but tedious. The result is best described using Dynkin diagrams.

Example: 

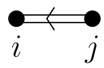
Rules:

- Each simple root $\alpha^{(i)}$, $i = 1, \ldots, r$ corresponds to a *vertex* (or node).

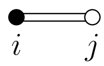
- Vertex (i) and vertex (j), where $i \neq j$, are connected by $|A^{ij}|$ lines (where $|A^{ij}|$ is the larger one of $|A^{ij}|$ and $|A^{ji}|$).



- If $A^{ij} = -1$ and $A^{ji} = -2$:



Alternative way to do this, is to use filled and open nodes, so the short one is filled:



Note:

$$A^{ij} = 2\,\frac{\alpha^{(i)} \cdot \alpha^{(j)}}{\alpha^{(j)} \cdot \alpha^{(j)}} = -1$$

$$A^{ji} = 2\,\frac{\alpha^{(j)} \cdot \alpha^{(i)}}{\alpha^{(i)} \cdot \alpha^{(i)}} = -2$$

Divide them:

$$2 = \frac{\alpha^{(j)} \cdot \alpha^{(j)}}{\alpha^{(i)} \cdot \alpha^{(i)}}$$

$$\Rightarrow \left|\alpha^{(j)}\right| > \left|\alpha^{(i)}\right|$$

So $\alpha^{(j)}$ is the long one.

Note: Renumbering the simple roots does not change the Dynkin diagram.

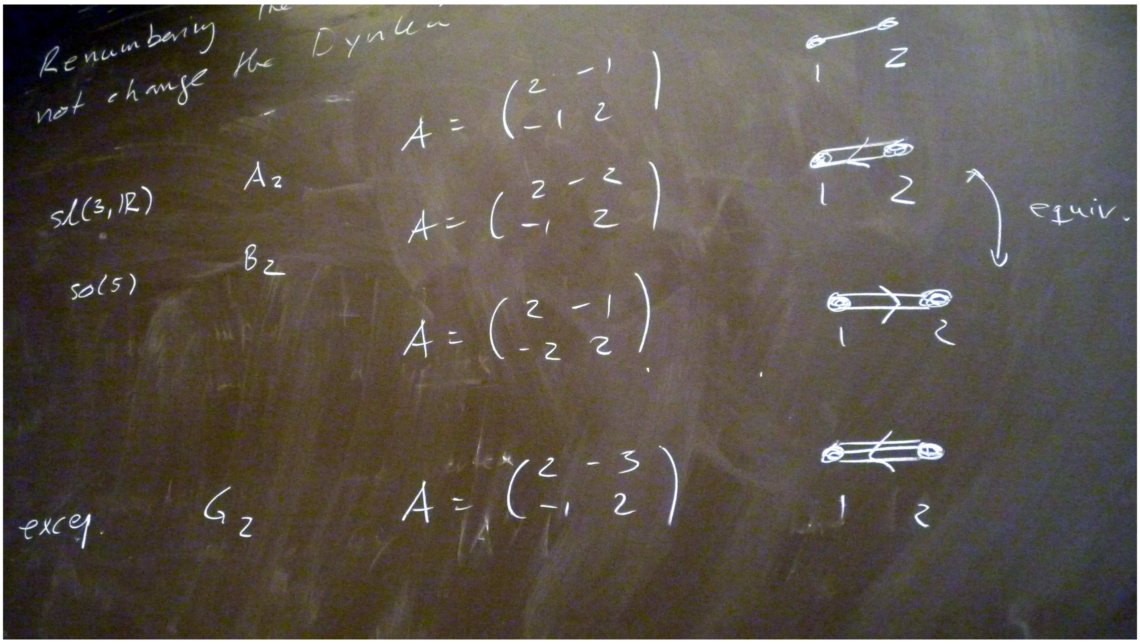


Figure 3. Examples.

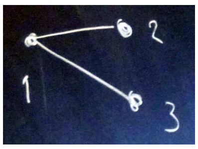
Example: Rank 3:

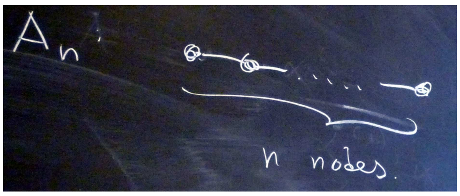
$$A = \begin{pmatrix} 2 & \alpha & \beta \\ a & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$$

The lower right matrix is just $A_1 \oplus A_1$. That's just two nodes without lines in between: $\bullet \quad \bullet$.

$$\det A = 2\,(4 - \alpha a - \beta b) > 0$$

$\det = 8$: $A_1 \oplus A_1 \oplus A_1$

$\det = 4$: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$  A_3



$\det = 2$: $\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ (figure 4)

Why aren't these (figure 4) the same?

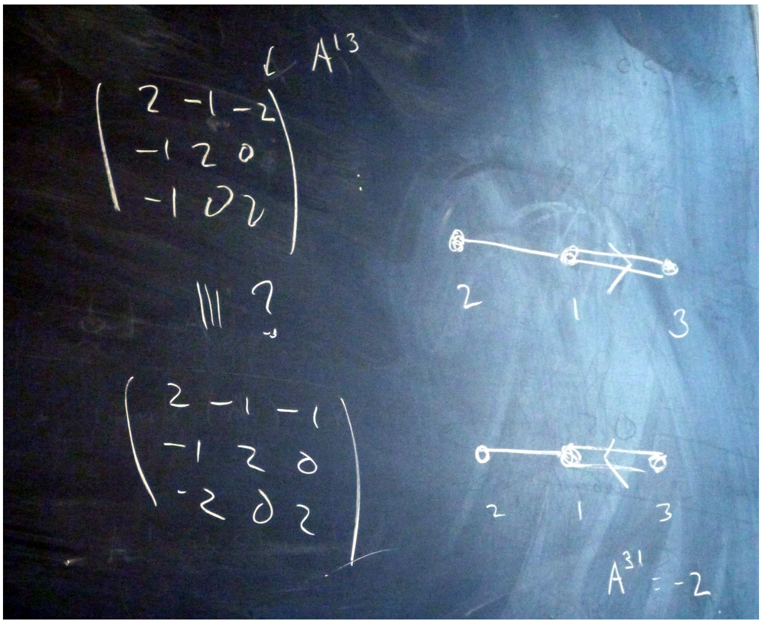


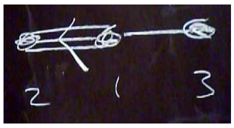
Figure 4. Why aren't these the same?

$$\det A = 0\colon \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \text{ (figure 4) } \; B_2^{(2)}\colon \text{affine}$$

(affinization of 2).

Also

$$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}\colon \quad \det = 0, \quad G_2^{(1)},$$



(affinization of 1).

7.4. Simple finite Lie algebras.

[We go through tables 4–8 from Chapter 7 of Fuchs and Schweigert (2003) *Symmetries, Lie Algebras and Representations — A graduate course for physicists*. That's pages 114, 115, 117, 118, 121 and 125; at the time of writing a subset thereof is available on Google Books Preview at http://books.google.com/books?id=B_JQryjNYyAC&lpg=PP1&pg=PA114.]

The ones with only one line are called *simply laced*.

ADE-class: simply laced: one line only in the Dynkin diagram. Appear in

- singularity classification
- modular invariants in conformal field theory
- QFT in 6 dimensions
- discrete subgroupos of SU(2).

The Dynkin diagram can also be used to describe representations: Highest weight: write on figure.

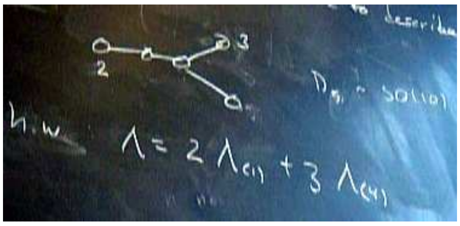


Figure 5. $D_5 \sim \mathrm{SO}(10)$. Highest weights written in the Dynkin diagram.

$$\Lambda = 2\Lambda_{(1)} + 3\,\Lambda_{(4)} = (2,0,0,3,0) \quad \text{in Slansky}$$

Highest root (highest weight of the adjoint)

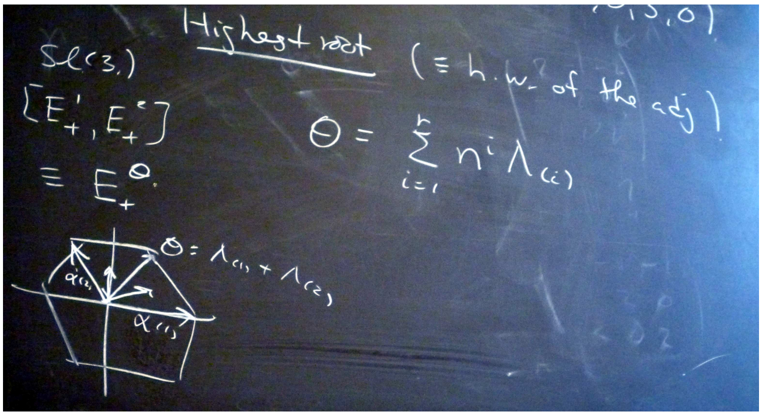


Figure 6.

$$\theta = \sum_{i=1}^{r} n^i \Lambda_{(i)} = \sum_{i=1}^{r} a_i\, \alpha^{(i)}$$

The a_i are positive integers called *Coxeter labels*.

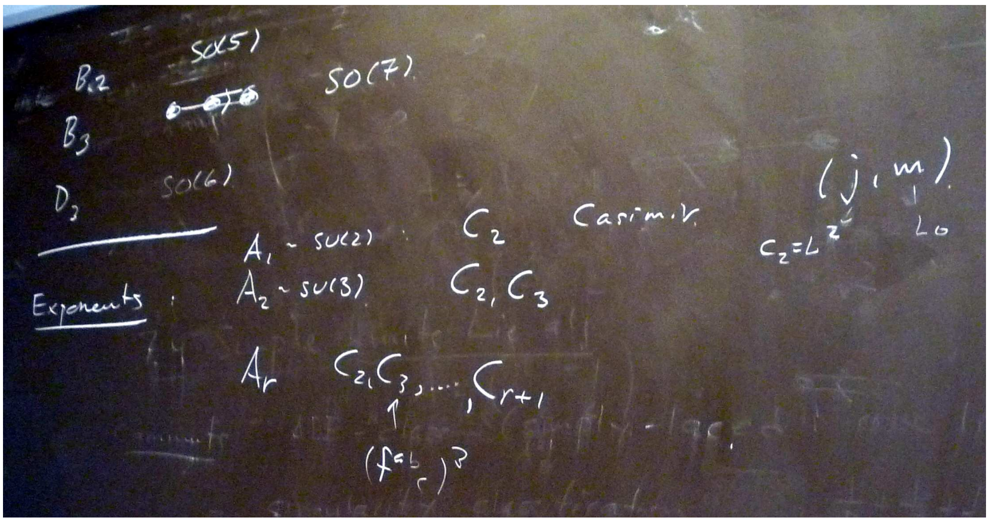


Figure 7. *Insert descriptive text here!* [Is this the right place to insert this figure? Hmm...]

7.7. Orthonormal basis

Finite Lie algebras, det $A>0$, symmetrizable. That we know from the definition of A.

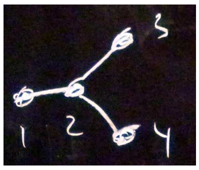
$$A^{ij}=2\,\frac{\alpha^{(i)}\cdot\alpha^{(j)}}{\alpha^{(j)}\cdot\alpha^{(j)}}$$

Multiply with something to get symmetric.

G^{ij} exists, metric on weight space. G^{ij} is non-degenerate, orthonormal basis!

Example: D_4 (SO(8)). The most symmetric one.

$$A_{D_4}=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & -1\\ 0 & -1 & 2 & 0\\ 0 & -1 & 0 & 2\end{pmatrix}$$



This is SO(8), I claim. The dimension of SO(8) is $\frac{8\times(8-1)}{2}=28$.

rank $=4\Rightarrow\dim\Phi=24\Rightarrow\dim\Phi_+=12$.

Roots:

$$\Phi(D_4)=\{(\pm1,0...,\pm1,...,0)\}\Rightarrow|\alpha|^2=2$$

(Two non-zero slots.)

How many are there? The signs give a factor 4, and then $4\times3/2$.

$$4\times\frac{4\times3}{2}=24.$$

Positive roots:

$$\Phi_+(D_4)=\{(0,...,0,+1,...,\pm1,...)\}$$

Simple ones: $(1,-1,0,0)$, $(1,0,-1,0)$ gives $A^{12}=+1$, no not possible.

Try $\alpha^{(1)}$ as $(1,-1,0,0)$ and α^2 as $(0,1,-1,0)$. That works, gives $A^{12}=-1$.

So we can use $(1,\,-1,\,0,\,0)$, $(0,\,1,\,-1,\,0)$, $(0,\,0,\,1,\,-1)$, $(0,\,0,\,1,\,1)$. All others are positive linear combinations of these four.

Hihest root:

$$\theta=\alpha^{(1)}+2\,\alpha^{(2)}+\alpha^{(3)}+\alpha^{(4)}=(1,1,0,0)$$

$$\sum_{i=1}^{r}n_i=\text{height}$$

$$\Lambda_R=\left\{m_i\alpha^{(i)}\colon m_i\in\mathbb{Z}\right\}$$

Check:

$$\sum_{i,j}m_i\alpha^{(i)}\cdot n_j\alpha^{(j)}=\sum_{i,j}m_in_j\,\alpha^{(i)}\cdot\alpha^{(j)}=$$

$$=\sum_i m_in_i\,2+2\sum_{i<j}m_i\,n_j\underbrace{\alpha^{(i)}\cdot\alpha^{(j)}}_{\in\{-1,0\}}\in2\mathbb{Z}$$

Vector representation (1, 0, 0, 0).

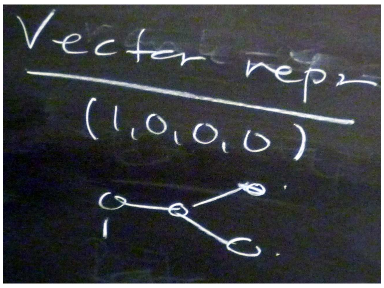


Figure 8.

$$\{(0, ..., \pm 1, ..., 0)\text{: only one non-zero entry}\} \quad \Rightarrow \dim = 8$$

Conjugacy class v, not a lattice: $\Lambda_{\mathrm{v}} = (1, 0, 0, 0) + \Lambda_R$

Spinor representation

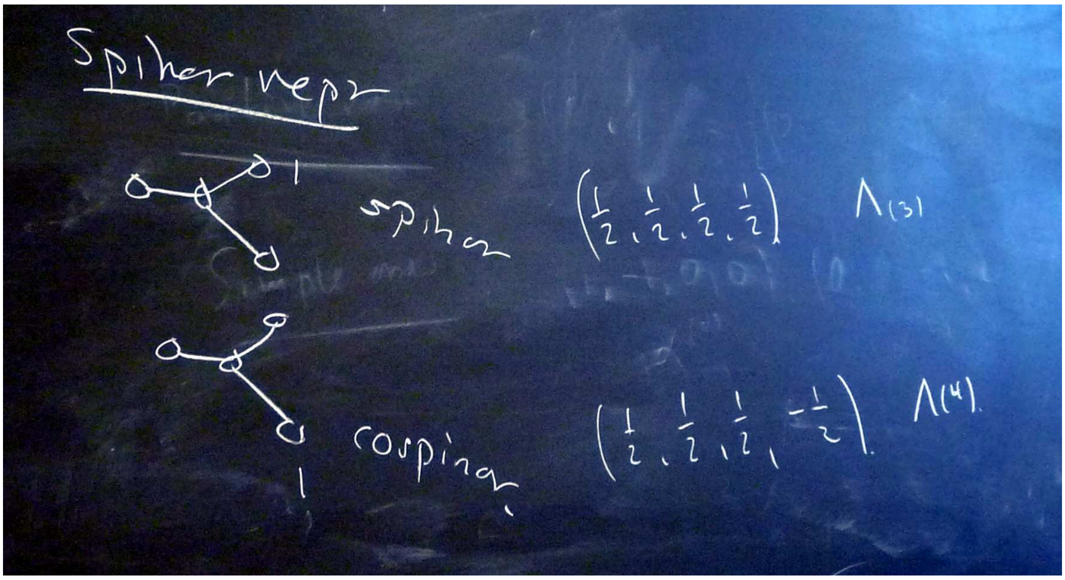


Figure 9.

Fundamental weights satisfy $\Lambda_{(i)} \cdot \alpha^{(j)} = \delta_i^j$.