2012–04–13 Lecturer: Bengt E W Nilsson

Chapter 6: General theory of Lie algebras and their representations.

The goal is to turn the results obtained for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ into a general theory and find the general classification (finite dimensional Lie algebras).

Definition. Lie algebra $\mathfrak{g}$

1) is a vector space whose elements are called generators of $\mathfrak{g}$,

2) $\exists$ a bilinear product called the Lie product such that $[x,x]=0 \forall x\in\mathfrak{g}$.

3) Jacobi idientity.

In a basis T^a of the Lie algebra $\mathfrak{g}$:

$$[T^a,T^b]=f^{a\,b}{}_c T^c,\quad a,b,c=1,\ldots,\dim\mathfrak{g}$$

$f^{a\,b}{}_c$ are called structure constants.

Example. Adjoint representation ad_x acts with the Lie bracket $[x,\cdot\,]$ on the representation space (module) which in this case is $\mathfrak{g}$ itself:

$$\mathrm{ad}_x\cdot y=[x,y]$$

$$\Rightarrow (T^a)^b{}_c = -f^{a\,b}{}_c$$

Exercise: Check this.

Definition. $\mathfrak{g}$ is *abelian* iff $[x,y]=0\forall x,y\in\mathfrak{g}$.

Definition. $\mathfrak{g}$ is *simple* iff $\mathfrak{g}$ has no (proper) invariant subalgebras (no proper ideals).

Definition. $\mathfrak{g}$ is *semi-simpole* iff $\mathfrak{g}=\sum_i\oplus\mathfrak{g}_i$ where all $\mathfrak{g}_i$ are simple.

Vector spaces V

Definition. Elements are called *vectors*. There exist addition and multiplication by scalar. For scalars $\alpha\in\mathbb{F}$ and vectors $v\in V$ we have distributivity:

1) $\alpha(v_1+v_2)=\alpha\,v_1+\alpha\,v_2$

2) $(\alpha+\beta)\,v=\alpha\,v+\beta\,v$

Definition. The *span* (or hull). Given a subset of vectors M, $\mathrm{Span}_{\mathbb{F}}(M)$ is a vector subspace of all linear combinations of these vectors.

Definition. A basis $\mathcal{B}$ is a subset of vectors in $\mathrm{Span}_{\mathbb{F}}(M)$ that span the whole subspace and is linear independent.

Definition. $d=\dim_{\mathbb{F}}V=|\mathcal{B}|$.

Definition. Direct sum $V_1\oplus V_2$ of vector spaces is such that

1) $\alpha(v_1\oplus v_2)=\alpha\,v_1\oplus v_2$

2) $(v_1\oplus v_2)+(w_1\oplus w_2)=(v_1+w_1)\oplus(v_2+w_2)$.

If $\mathcal{B}_1=\{\hat{v}_i\}$ and $\mathcal{B}_2=\{\hat{w}_i\}$, then a basis in $V_1\oplus V_2$ is

$$\mathcal{B}=\{\hat{v}_i\oplus 0\}\cup\{0\oplus\hat{w}_a\}$$

$$|\mathcal{B}|=|\mathcal{B}_1|+|\mathcal{B}_2|$$

$$d=d_1+d_2$$

EXAMPLE. A vector in D dimensions can be decomposed into vectors in two subspaces ($d = d_1 + d_2$).

$$A_M = (A_\mu, A_m)$$

EXAMPLE. What about the metric?

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & g_{\mu n} \\ g_{m\nu} & g_{mn} \end{pmatrix}$$

Which vector space are $g_{m\nu}$ and $g_{\mu n}$ in? One index belongs to V_1, one index belongs to V_2. $g_{\mu\nu}$: $V_1 \otimes V_2$.

First. Cartesian (or Kronecker) product

$$V_1 \times V_2 = \{(v_1, v_2) \colon v_1 \in V_1, v_2 \in V_2\}$$

Ordered pairs.

$g_{\mu n}$ has properties $\alpha(g_{\mu n} + g'_{\mu n}) = \alpha\, g_{\mu n} + \alpha\, g'_{\mu n}, (\alpha + \beta) g_{\mu n} = \alpha g_{\mu n} + \beta\, g_{\mu n} \Rightarrow g_{\mu n} \in V_1 \otimes V_2$.

So $\otimes$ is obtained from $\times$ by demanding that $(\xi\, v_1, v_2)$ is identified with $(v_1, \xi v_2)$ since this eliminates $(0, v_2)$ and $(v_1, 0)$.

Lie algebras: The Cartan–Weyl basis.

- Gives the mathematical foundation for further analysis of Lie algebras.
- A step towards the classification.
- Later: general analysis of Lie algebras: Levi's theorem.

Goal is to understand why, and if, one can always formulate all properties of a simple Lie algebra in terms of only A^{ij}, the Cartan matrix.

Recall $\mathfrak{sl}(3)$:

1)

$$[H^i, E_\pm^j] = \pm A^{ji} E_\pm^j$$

$$A^{ji} \equiv (\alpha^{(j)})^i$$

2)

$$\alpha^{(i)} \cdot \alpha^{(j)} = A^{ij}, \quad G^{ij} = A^{ij}$$

3)

$$\mathrm{tr}(H^i H^j) = A^{ij}$$

Killing form κ^{ij}.

Also in fact (4) $[E_+^1, [E_+^1, E_+^2]] = 0$, i.e. $(\mathrm{ad}_{E_+^1})^{1 - A^{21}} E_+^2 P = 0$. Serre relation.

6.1. Cartan subalgebras.

We will describe Lie algebras by using different bases:

1) a general one is Cartan–Weyl.

2) at the oend of the day we define *Chevalley* basis.

The first step will be to identify the *Cartan* algebra $\mathfrak{g}_0$.

$$\mathfrak{g}_0 \equiv \mathrm{Span}_{\mathbb{C}}\{H^i : i = 1, \ldots, r\} \quad (r = \mathrm{rank})$$

such that r is the maximal number of commuting elements in $\mathfrak{g}$: $[H^i, H^j] = 0 \forall i, j$.

$\Rightarrow$All elements H^i in g_0 are simultaneously diagonalizable, i.e.

$$\Bigg[\underbrace{x}_{\in\mathfrak{g}_0}, \underbrace{\tilde{T}^a}_{\mathfrak{g}\text{ not in }\mathfrak{g}_0}\Bigg] = f^{x a}{}_b \tilde{T}^b = f^x \delta^a{}_b \tilde{T}^b$$

secular equation

$$\det\left((f^x)^a{}_b - f^x \delta^a{}_b\right) = 0$$

For this equation to have d solutions, which means that we must allow for complex numbers.

6.2. Roots (Note: we don't assume any relation between $\left(\alpha^{(j)}\right)^i$ and A^{ji} here.)

The previous relation is often written

$$[h, y] = \alpha_y(h)\, y$$

$h \in \mathfrak{g}_0$, $y \in \mathfrak{g}$. $\alpha_y(h) = \mathrm{root}$. $\alpha_y(h)$ depneds linearly on h:

$$\alpha_y : g_0 \to \mathbb{C} \quad (\text{linear map})$$

$\Rightarrow \alpha_y \in \mathfrak{g}^*$ (the dual space to $\mathfrak{g}_0 \equiv$ the set of all linear maps from $\mathfrak{g}_0$ to $\mathbb{C}$).

$\left(\alpha^{(j)}\right)^i, h = h_i H^i$ where H^i is the basis.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\underbrace{\oplus_{\alpha \neq 0}}_{\text{roots}} \mathfrak{g}_\alpha\right)$$

$\mathfrak{g}_0$ – Cartan – r-degenerated. $\mathfrak{g}_\alpha$ — elements in $\mathfrak{g}$ associated with root α (non-degenerated).

Root system = {all roots $(\alpha \neq 0)$} $= \Phi = \Phi(\mathfrak{g})$

Cartan–Weyl basis

$$\mathcal{B} = \{H^i : i = 1, \ldots, r\} \cup \{E^\alpha : \alpha \in \Phi\}$$

with

$$[H^i, H^j] = 0, \quad [H^i, E^\alpha] = \alpha^i E^\alpha, \quad [E^\alpha, E^\beta] = e_{\alpha+\beta} E^{\alpha+\beta}$$

6.3. Killing form

In $\mathfrak{sl}(2, \mathbb{R})$ we use $T^a = (H, E_+, E_-)$. In the 2-dimensional representation:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Tr}_{(2)}(T^a T^b) = \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{array}{l} H \\ E_+ \\ E_- \end{array} \\ \begin{array}{ccc} H & E_+ & E_- \end{array} \end{array}$$

In the three-dimensional representation:

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad E_+^2 = 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{etc}$$

$$\mathrm{Tr}_{(3)}(T^a T^b) = 4\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition. Representation independent definition of Killing form.

Usae the ad_x. Try:

$$\mathrm{tr}(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}) = ?$$

To get the relevant matrix let it act on some T^c:

$$\mathrm{ad}_{T^a} \circ \underbrace{\mathrm{ad}_{T^b}(T^c)}_{f^{bc}{}_d T^d} = \underbrace{f^{bc}{}_d f^{ad}{}_e}_{(M^{ab})^c{}_e} T^e$$

$$\mathrm{Tr}(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}) = \mathrm{Tr}\, M^{ab} = (M^{ab})^c{}_c = f^{bc}{}_d f^{ad}{}_c$$

For $\mathfrak{sl}(2)$

$$\begin{cases} [H, H] = 0 \\ [H, E_\pm] = \pm 2E_\pm \\ [E_+, E_-] = H \end{cases} \qquad \begin{cases} f^{hh}{}_a = 0 \\ f^{h\pm}{}_\pm = 2 \\ f^{+-}{}_\pm = 0 \\ f^{+-}{}_H = 1 \end{cases}$$

$$\mathrm{Tr}(\mathrm{ad}_H \circ \mathrm{ad}_H) = f^{Ha}{}_b f^{Hb}{}_a = 8$$

$$\mathrm{Tr}(\mathrm{ad}_{E_+} \circ \mathrm{ad}_{E_-}) = f^{+a}{}_b f^{-b}{}_a = 4$$

$$\mathrm{Tr}(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}) = 4\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4: index for ad. The matrix is the Killing form.

Recall for the 2-dimensional we found exactly

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

i.e. its index = 1.

Metric on $\mathfrak{g}$. In general

$$\kappa^{ab} = \begin{pmatrix} \kappa^{ij} & 0 \\ 0 & \text{root part} \end{pmatrix}$$

Properties: κ is *invariant* under the $\mathfrak{g}$.

$$\kappa(x,y)\colon \kappa([x,y],z)=\kappa(x,[y,z])$$

Proof: EXERCISE! Write out the two sides in terms of ad's and use the trace.

6.4. Properties of roots and root systems

Roots span $\mathfrak{g}_0^*$

Each root space $\mathfrak{g}_\alpha$ is r-dimensional

For each α only $\pm\alpha$ are roots. (i.e. 2α is never a root! $[E^\alpha,E^\alpha]=0$)

α^i are all real numbers! In fact: integers (see construction of $\mathfrak{sl}(3)$ from two $\mathfrak{sl}(2)$'s).

6.5. Structure of the Cartan–Weyl basis

So far:

$$[H^i,H^j]=0. \quad i=1,\ldots,r$$

$$[H^i,E^\alpha]=\alpha^i E^\alpha, \quad \alpha\in\Phi$$

$$\Rightarrow [H^i,[E^\alpha,E^\beta]]=[\text{Jacobi}]=(\alpha^i+\beta^i)[E^\alpha,E^\beta]$$

Three cases for $\alpha^i+\beta^i$:

1) if $\alpha^i+\beta^j\neq 0\Rightarrow [E^\alpha,E^\beta]\sim E^{\alpha+\beta}$ i.e. $[E^\alpha,E^\beta]=e_{\alpha,\beta}E^{\alpha+\beta}\Rightarrow \alpha+\beta\in\Phi$.

2) if $\alpha^i+\beta^i=0\Rightarrow [E^\alpha,E^{-\alpha}]=\sum_{i=1}^r \hat{\alpha}_i H^i\in\mathfrak{g}_0$

3) $[E^\alpha,E^\beta]=0\Rightarrow \alpha+\beta\notin\Phi$.

Freedom left to use: $e_{\alpha,\beta}$ and $\hat{\alpha}_i$ not specified and basis in $\mathfrak{g}_0$ not chosen.

Note: these choices will affect the Killing form.

In terms of $f^{ab}{}_c$:

$$\begin{array}{ccc} \mathfrak{g} & \mathfrak{g}_0 & \Phi \\ \downarrow & \downarrow & \downarrow \\ a \;= & (i, & \alpha) \end{array}$$

$$f^{ij}{}_k=0, \quad f^{i\alpha}{}_\beta=\alpha^i\,\delta^\alpha{}_\beta$$

$$f^{\alpha\beta}{}_i=\hat{\alpha}_i\delta_{\alpha,-\beta}, \quad f^{\alpha i}{}_j=0$$

$$f^{\alpha\beta}{}_\gamma=\begin{cases} e_{\alpha,\beta}\,\delta_{\alpha+\beta,\gamma} & \text{if } \alpha+\beta\in\Phi \\ 0 & \text{if } \alpha+\beta\notin\Phi \end{cases}$$

Note:

$$\kappa^{ij}\sim\sum_{a,b} f^{ia}{}_b f^{jb}{}_a=\sum_{\alpha,\beta} f^{i\alpha}{}_\beta f^{j\beta}{}_\alpha=\sum_\alpha \alpha^i\,\alpha^j$$

Exercise: Check that this gives the right answer.

6.6. Positive roots

Split $\mathfrak{g}$ as

$$\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_0\oplus g_-$$

Gauss decomposition, or triangular decomposition. $\mathfrak{g}_+$ positive roots, $\mathfrak{g}_-$ negative roots.

6.7. Simple roots

Simple roots is a set of positive (r of them) such that

- they are positive
- all other positive roots are linear combinations with non-negative (integer) coefficients.

Note: if $\alpha^{(i)}$ and $\alpha^{(j)}$ $(j \neq i)$ are simple, then $\alpha^{(i)} - \alpha^{(j)}$ is never a root (can't be a negative root, can't be a positive root).

6.8. Chevalley basis

Start from the completely general algebra in Cartan–Weyl basis

$$\left[H^i, E^j_\pm\right] = \pm\left(\alpha^{(j)}\right)^i E^j_\pm$$

$$\mathrm{tr}(H^i H^j) = \kappa^{ij}$$

$$\alpha^{(i)} \underset{\substack{\uparrow \\ \kappa}}{\cdot} \alpha^{(j)} = G^{ij}$$

To go from here we can define coroots

$$\check{\alpha}^{(i)} \equiv \frac{\alpha^{(i)}}{C^{(i)}}$$

where $C^{(i)}$ is some number.

Tus we have the possibility to define coroots all of the same length (not the case for the roots). Also, from $\check{\alpha}$ we can define its dual vector space, the weight space.

So $\check{\alpha}^{(i)} \cdot \check{\alpha}^{(j)} = \check{G}^{ij}$, $\Lambda_{(i)} \cdot \check{\alpha}^{(j)} = \delta^i{}_j, \Lambda_{(i)} \cdot \Lambda_{(j)} = \check{G}^{-1}_{ij}$ and $\check{\alpha}^{(i)} = \check{G}^{ij}\, \Lambda_{(j)}$. All using the natural metric κ^{ij}.

Now we can always use a basis where $\left(\check{\alpha}^{(i)}\right)^j$ and $(\kappa^{-1})_{ij}$ are determined by $\check{G}^{ij}$!

$$\begin{cases} \left(\check{\alpha}^{(i)}\right)^j = \check{G}^{ij} \\ (\kappa^{-1})_{ij} = (\check{G}^{-1})_{ij} \end{cases}$$

Then

$$\check{\alpha}^{(i)} \underset{\substack{\uparrow \\ \kappa^{-1}}}{\cdot} \check{\alpha}^{(j)} = \check{G}^{ij}$$

How is this done? i.e. relating $\check{\alpha}$ to κ?

Define $H^{(i)} \equiv H^j \kappa_{jk} \left(\check{\alpha}^{(i)}\right)^k$. But then

$$\mathrm{tr}\, H^{(i)}\, H^{(j)} = \underbrace{\mathrm{tr}\, H^k\, H^l}_{\kappa^{kl}} \cdot \kappa^{-1}_{km} \left(\check{\alpha}^{(i)}\right)^m \kappa^{-1}_{ln} \left(\check{\alpha}^{(j)}\right)^n =$$

$$= \check{\alpha}^{(i)} \underset{\substack{\uparrow \\ \kappa}}{\cdot} \check{\alpha}^{(j)} = \check{G}^{ij}$$

Next step:

$$\left[H^{(i)}, E_{\pm}^{j}\right] = \pm\left(\alpha^{(j)}\right)^{(i)} E_{\pm}^{j}$$

where

$$\left(\alpha^{(j)}\right)^{(i)} = \left(\alpha^{(j)}\right)^{m} \kappa_{mn}^{-1} \left(\check{\alpha}^{(i)}\right)^{n} = \alpha^{(j)} \cdot \check{\alpha}^{(i)}$$

Next: Define the Cartan matrix

$$A^{ji} = \alpha^{(j)} \cdot \check{\alpha}^{(i)}$$

Finally, choose C_{α} so that the $A^{ii} = 2$ for all i.

$$\check{\alpha}^{(i)} = \frac{2\,\alpha^{(i)}}{\alpha^{(i)} \cdot \alpha^{(i)}}$$