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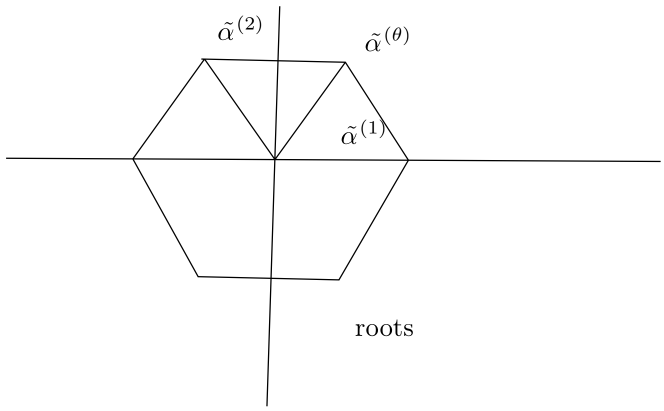
Back to $\mathfrak{sl}(3,\mathbb{C})$: A_2 in Cartan's classification.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

And we have this fantastic picture, which is very hard to draw.



roots

$\alpha^{(1)} = (2,-1); \alpha^{(2)} = (-1,2)$. In this basis here $\alpha^{(i)} \cdot a^{(j)} = A$, the metric $G^{ij} = A^{ij}$. When these are drawn in the diagram we need a trivial metric δ^{ij}. When we go to this basis we call them

$$\tilde{\alpha}^{(1)} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\alpha}^{(2)} = \sqrt{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

$$\left(\tilde{\alpha}^{(1)}\right)^2 = \left(\tilde{\alpha}^{(2)}\right)^2 = 2$$

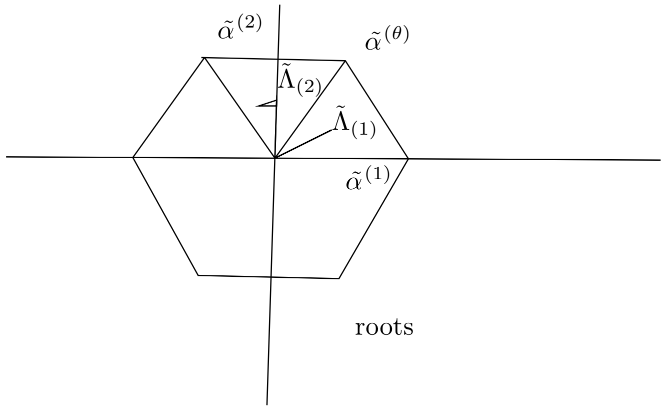
A_2 has six roots, plus two Cartan generators $\Rightarrow \dim = 8$. The Cartan generators sit in the middle of the diagram here, in some sense.

$$\left[H^i, E_{\pm}^{(j)}\right] = \pm A^{ji} E_{\pm}^{(j)}$$

Representations: In two steps: (1) Extend the root diagram to a lattice spanned by two vectors $\tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)}$. All linear combinations with integer coefficients. Then (2) construct the dual lattice to the root lattice, which is called the weight lattice: Find two vectors $\Lambda_{(i)}$ that span this vectorspace such that $\tilde{\Lambda}_{(i)} \cdot \tilde{\alpha}^{(j)} = \delta_i{}^j$, i.e. $\left(\tilde{\Lambda}_{(i)}\right)_k \left(\tilde{\alpha}^{(j)}\right)^k = \delta_i{}^j$. Solve for $\tilde{\Lambda}_{(i)}$:

$$\tilde{\Lambda}_{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \tilde{\Lambda}_{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{2}{\sqrt{3}} \end{pmatrix}$$

$$\left(\tilde{\Lambda}_{(1)}\right)^2 = \left(\tilde{\Lambda}_{(2)}\right)^2 = \frac{2}{3}$$



Now note $\tilde{\alpha}^{(i)} \cdot \tilde{\alpha}^{(j)} = G^{ij}$, $\tilde{\Lambda}_{(i)} \cdot \tilde{\Lambda}_{(j)} = (G^{-1})_{ij}$. Note

$$\left|\tilde{\alpha}^{(1)} \times \tilde{\alpha}^{(2)}\right| = \sqrt{3}, \quad \left|\tilde{\Lambda}_{(1)} \times \tilde{\Lambda}_{(2)}\right| = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{\text{Vol(root)}}{\text{Vol(weight)}} = 3$$

$$\Rightarrow \quad \text{Number of conjugacy classes} = 3$$

Highest weight (hw, v_Λ, where Λ has to do with the representation)

$$\begin{cases} E_+^i v_\Lambda = 0 \\ H v_\Lambda = \Lambda\, v_\Lambda \end{cases}$$

The rest of the representation (vector in the module) are obtained by acting with E_-^i:

$$\begin{cases} E_-^i v_\Lambda \\ E_-^i E_-^j v_\Lambda \\ \vdots \end{cases}$$

Note: $E_+^i E_-^j v_\Lambda$ are irrelevant here since this is equal to $\left[E_+^i, E_-^j\right] v_\Lambda + E_-^j \cancel{E_+^i v_\Lambda}$ so E_+'s can always be eliminated from the expression, by using the algebra.

Example. Let $\Lambda = \Lambda_{(1)}$. Then $H^i v_{\Lambda_{(1)}} = \Lambda_{(1)}^i\, v_{\Lambda_{(1)}}$.

$$\left[H^i, E_\pm^j\right] = \pm \underbrace{\left(\alpha^{(j)}\right)^i}_{A^{ji}} E_\pm^j$$

$$H^i\left(E_-^j\, v_{\Lambda_{(1)}}\right) = \left[H^i, E_-^j\right] v_{\Lambda_{(1)}} + E_-^j H^i\, v_{\Lambda_{(1)}} = -\left(\alpha^{(j)}\right)^i E_-^j\, v_{\Lambda_{(1)}} + (\Lambda_{(1)})^i E_-^j\, v_{\Lambda_{(1)}} =$$

$$= \left((\Lambda_{(1)})^i - \left(\alpha^{(j)}\right)^i\right)\left(E_-^j V_{\Lambda_{(1)}}\right)$$

Acting on the highest weight:

$$E^1_-: \quad \Lambda_{(1)} \to \Lambda_{(1)} - \alpha^{(1)}$$

$$E^2_-: \quad \Lambda_1 \to \Lambda_{(1)} - \alpha^{(2)}$$

So $E^i_- v_{\Lambda_{(1)}} = v_{\Lambda_{(1)} - \alpha^{(i)}}$.

$$E^i_- E^j_- v_{\Lambda_{(1)}} = v_{\Lambda_{(1)} - \alpha^{(i)} - \alpha^{(j)}}$$

Does it stop? (i.e. is the representation finite-dimensional?)

Note: $\alpha^{(1)} = 2\,\Lambda_{(1)} - \Lambda_{(2)} = A^{1i}\Lambda_{(i)}$ and $\alpha^{(2)} = 2\,\Lambda_{(2)} - \Lambda_{(1)} = A^{2i}\Lambda_{(i)}$.

$$\Lambda_{(1)} - \alpha^{(1)} - \alpha^{(2)} = -\Lambda_{(2)}$$

The three weights $\otimes$ (triangle with point down) is the representation **3**.

Starting with $\Lambda_{(2)}$ (triangle with point up) we get $\bar{\mathbf{3}}$.

Starting from $\Lambda_{(1)} + \Lambda_{(2)} = \alpha^{(\theta)}$ i.e. the adjoint representation. 8-dimensional.

So highest weight are written

$$\Lambda = n^1 \Lambda_{(1)} + n^2 \Lambda_{(2)}, \quad n^i \in \mathbb{Z}$$

if n^i are non-negative this is a highest weight.

In the footnote under the table: $6 = (20)$. (n^1, n^2): Dynkin labels.

Definition of positive, negative and simple roots. By choosing two of the root vectors as basis vectors we can expand all other roots in this basis. By choosing the two roots as "far as possible away from each other" (maximal angle <180°) — example: $\alpha^{(1)}$ and $\alpha^{(2)}$ in $\mathfrak{sl}(2, \mathbb{C})$ — then half of the roots will have positive first non-zero component and the other half negative.

A_2: In the root basis the positive roots are: $\alpha^{(1)} = (1,0), \alpha^{(2)} = (0,1), \alpha^{(\theta)} = (1,1)$. The negative roots are $-\alpha^{(1)} = (-1, 0)$, $-a^{(2)} = (0, -1)$, $-\alpha^{(\theta)} = (-1, -1)$. The two roots used to distinguish positive and negative, $\alpha^{(1)}$ and $\alpha^{(2)}$, are called *simple roots*.

Note: When determining the dimension of a representation one must know the dimension of the space at each weight!

To determine the dimension of the space at each weight one can use the Freudenthal's recursion formula. Multiplicity of representation Λ, with λ being one of the wieghts in Λ:

$$\mathrm{mult}_\Lambda(\lambda) = \frac{2\displaystyle\sum_{\alpha>0}\sum_{m>0} (\lambda + m\,\alpha, \alpha)\,\mathrm{mult}_\Lambda(\lambda + m\,\alpha)}{(\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho)}$$

Define the multiplicity $\mathrm{mult}_\Lambda(\Lambda) = 1$, then the $\mathrm{mult}_\lambda(\lambda)$ for all λ's follow. Here

$$\rho = \sum_{\substack{\alpha>0 \\ \text{positive} \\ \text{roots}}} \frac{1}{2}\alpha$$

In A_2:

$$\rho = \frac{1}{2}\big(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(\theta)}\big) = \alpha^{(1)} + \alpha^{(2)}$$

This is called the Weyl vector. Very important vector in this business.

Example in adjoint of A_2:

$$\Lambda+\rho=2\big(\alpha^{(1)}+\alpha^{(2)}\big)$$

$$(\Lambda+\rho,\Lambda+\rho)=4\left(\alpha^{(1)}+\alpha^{(2)},\alpha^{(1)}+\alpha^{(2)}\right)=4(2+2-1-1)=8$$

Set $\lambda=\alpha^{(1)}\Rightarrow$

$$(\lambda+\rho,\lambda+\rho)=6$$

$$\Rightarrow\text{denominator}=8-6=2$$

$$\text{mult}_{\Lambda}\big(\lambda=\alpha^{(1)}\big)=...=1$$

$$\text{mult}_{\Lambda}(\lambda=0)=2.$$

OK. That's the Cartan algebra that sits in the origin.

Tensor product

In A_2 we talked about $T_{ij}=T_{(ij)}+T_{[ij]}$. No Kronecker here, so you can't trace it.

$$3\times3=\underbrace{\frac{3\times4}{2}}_{(2,0)}+\frac{3\times2}{2}$$

$$T_{[ij]}=\varepsilon_{ijk}T^k\text{:}\quad\bar{3}$$

$$3\otimes3=6\oplus\bar{3}$$

Conjugacy clases

Root lattice Λ_R. Weight lattice $\Lambda_R^*=\Lambda_W$

$$\Lambda_W=\Lambda_R\cup(\Lambda_R+\Lambda_{(1)})\cup(\Lambda_R+\Lambda_{(2)})$$

$$\text{index}=n^1+2\,n^2(\text{mod}\,3)$$

	Λ	index
adjoint	$(1,1)$	0
3	$(1,0)$	1
$\bar{3}$	$(0,1)$	2
6	$(2,0)$	2
$\bar{6}$	$(0,2)$	1