Group theory 2012-03-21 Lecturer: Bengt Nilsson

Recall:

Put together two $\mathrm{sl}(2,\mathbb{R})$'s to get an even larger algebra:

$$\begin{aligned}\left[H^i, E_{\pm}^j\right] &= A^{ji} E_{\pm}^j \quad i,j=1,2\\ [E_{\pm}^1, E_{\pm}^2] &= E_{\pm}^{\theta}\\ \left[E_{\pm}^i, E_{\mp}^j\right] &= \delta^{ij} H^j\end{aligned}$$

So with $A = \left(\begin{smallmatrix} 2 & 0\\ 0 & 2\end{smallmatrix}\right)$ then the new $g = \mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$ (dim=6).

A is called the Cartan matrix. For other such A's we have:

$$\begin{aligned}\begin{pmatrix} 2 & -1\\ -1 & 2\end{pmatrix} &\Rightarrow g = \mathrm{sl}(3,\mathbb{R}) \quad : \quad A_2 \quad \dim = 8 = 6+2\\ \begin{pmatrix} 2 & -2\\ -1 & 2\end{pmatrix} &\Rightarrow g = \mathrm{so}(5) \quad : \quad B_2\\ \begin{pmatrix} 2 & -3\\ -1 & 2\end{pmatrix} &\Rightarrow g = - \quad : \quad G_2\end{aligned}$$

The last one is not a matrix Lie algebra.

In the Cartan matrix the first row is a vector $\tilde{\alpha}^{(1)}$ and the second row $\tilde{\alpha}^{(2)}$.

We used the metric $G^{ij} = \mathrm{Tr}(H^i H^j) = \left(\begin{smallmatrix} 2 & -1\\ -1 & 2\end{smallmatrix}\right) = A$, to be able to draw the root diagram in a symmetric fashion. We diagonalised this metric to be able to draw the diagram in an orthonormal basis.

In an orthonormal basis we found

$$\begin{aligned}\tilde{\alpha}^{(1)} &= \sqrt{2}\begin{pmatrix} 1\\ 0\end{pmatrix}\\ \tilde{\alpha}^{(2)} &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1\\ \sqrt{3}\end{pmatrix}\end{aligned}$$

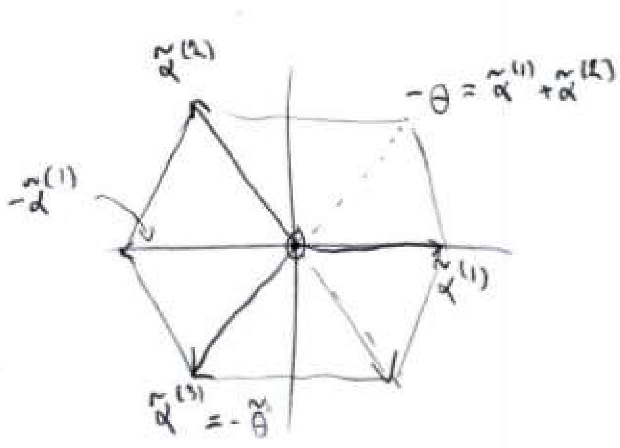

Figure 1.

$$\tilde{\alpha}^{(i)} \cdot \tilde{\alpha}^{(j)} = A^{ij}$$

The $\pm\tilde{\alpha}^{(i)}, \quad i=1,2,3$ are called the **roots** of A_2. Here A_2 refers to all of $\mathrm{sl}(3,\mathbb{R}), \mathrm{SU}(3), \mathrm{sl}(3,\mathbb{C})$.

The diagram above is called a **root diagram**.

B_2

Now we drop the ˜!

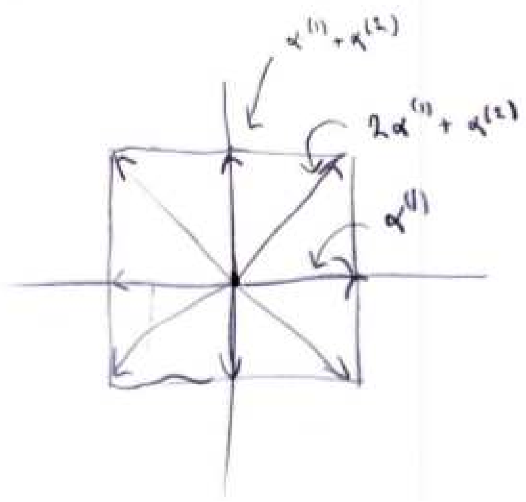

Figure 2.

All the roots are $= p\alpha^{(1)} + q\alpha^{(2)}$.

Note that the Cartan matrix here

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

Is not symmetric, so $\alpha^{(1)}$ and $\alpha^{(2)}$ does not have the same length.

Cartan classification

Classical matrix algebras

$$\begin{array}{cc} A_n & \mathrm{sl}() \\ B_n & \mathrm{so(odd)} \\ C_n & \mathrm{sp}() \\ D_n & \mathrm{so(even)} \end{array}$$

Exceptional

$$\begin{array}{c} G_2 \\ F_4 \\ E_6 \\ E_7 \\ E_8 \end{array}$$

All these have $\det A > 0 \Leftrightarrow$finite dimensional Lie algebras.

$\det(A) = 0 \quad \Leftrightarrow$ infinite-dimensional.

$\det(A) < 0 \quad \Leftrightarrow$ Kac-Moody,Hyperbolic,Lorentzian etc.

If A is different type from above we have for example Bocherd,.....

***G*$_2$**

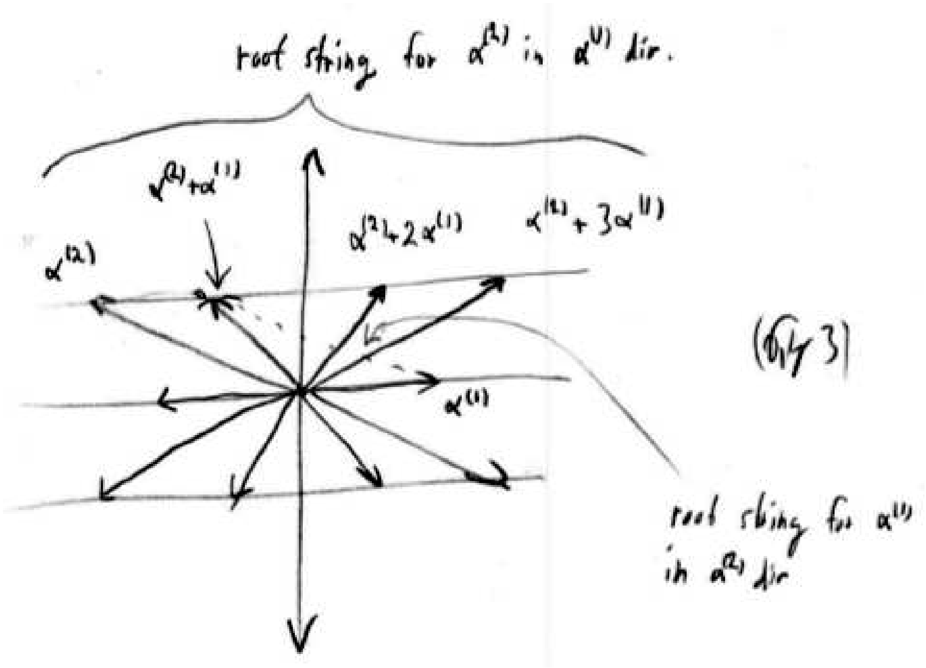

Figure 3.

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Representations of A_2

Before looking at the general theory we discuss *tensors from indices*:

Two topics:

1. Tensor products

2. Decompositions

1) Tensor products

In physics e.g. elem. part. transform according to some representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$.

quarks	3	of SU(3)
antiquarks	$\bar{3}$	of SU(3)
gluons	8	of SU(3)
hadrons	$8, 10, \ldots$	of SU(3)

The 8 above is related to the root diagram.

One thing many does now is looking at SU(N) where $N \to \infty$, called the $\frac{1}{N}$ expansion.

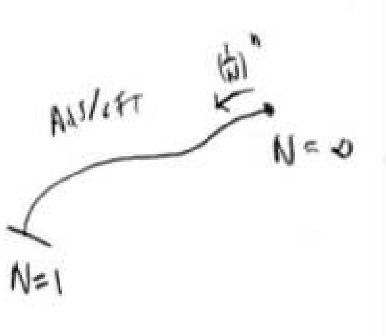

Figure 4.

Exercise in indices:

SO(3): The invariant tensors are $\delta_{ij}, \varepsilon_{ijk}$.

$$\begin{aligned} T_{ij} &\to T'_{ij} = g_{ik} g_{jl} T_{kl} \\ \delta_{ij} &\to \delta'_{ij} = g_{ik} g_{jl} \delta_{kl} \\ &= (g g^T)_{ij} \\ &= \delta_{ij} \end{aligned}$$

Where the last step is by definition of SO(N), they are matrices such that $g g^T = I$.

Infinitesimal (i.e. in the Lie algebra so(3)):

$$\delta(T_{ij}) = g_{ik} g_{jl} T_{kl} - T_{ij}$$

if we now expand $g = e^{\alpha^i \Lambda^i}$, we get

$$\delta(T_{ij}) = \alpha^i (\Lambda^i)_{ik} T_{kj} + \alpha^i (\Lambda^i)_{jk} T_{ik}$$

We have generators $g = -\varepsilon^{ijk}$.

Exercise 1. Show that ε^{ijk} is an inv tensor of the Lie algebra!

Now repr 3 of SO(3): T_i, this is just a vector.

Now we can take the tensor product $3 \otimes 3$: T_{ij}

From QM we know $3 \otimes 3 = 5 \oplus 3 \oplus 1$.

$$T_{ij} = T_{[ij]} + T_{\widetilde{(ij)}} + \delta_{ij} T_{kk}$$

Where the first term is antisymmetrisation, the second is symmetrisation with trace removed, and the third term is the trace.

For a three index tensor T_{ijk}

$$T_{ijk} = T_{[ijk]} + \ldots$$

But now $[ijk]$ is not irreducible because we have the invariant ε_{ijk}. But already here it starts to get hairy! We need Young tableux etc but with finite group theory knowledge it gets easier.

sl(3, ℝ)(or SU(3))

Invariant tensor is just ε^{ijk}.

$$T_i \;\Rightarrow\; T_{ij} = T_{[ij]} + T_{(ij)}$$

Here we cannot remove the trace since δ is not an invariant!

$$T_{[ij]} \;=\; \varepsilon_{ijk} T^k \quad \leftarrow \mathrm{repr}\,\bar{3}$$

Which gives us

$$3 \otimes 3 \;=\; \bar{3} \otimes 6$$

Also we have $T_i{}^j$ which corresponds to $3 \otimes \bar{3} = 8 \oplus 1$:

$$T_i{}^j \;=\; \delta_i{}^j T + \tilde{T}_i{}^j$$

Here we have a δ since this just corresponds to the.

Decomposition

Can we decompose representations under $\mathrm{sl}(3,\mathbb{R})$ to representations under the subgroup $\mathrm{sl}(2,\mathbb{R})$?

Example: 3 of $\mathrm{sl}(3,\mathbb{R})$

$$\begin{pmatrix} \mathrm{sl}(2,\mathbb{R}) & & 0 \\ & & 0 \\ 0 & 0 & . \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

So we have $3 \rightarrow 2 \oplus 1$.

Indices:

$$\begin{aligned} T_i \;&:\; i = 1, 2, 3 = (a, 3), \quad a = 1, 2 \\ T_i \;&\rightarrow\; T_a \oplus T_3 \end{aligned}$$

Ex:

$$\begin{aligned} T_i{}^j \;&=\; T_a{}^b \oplus T_a{}^3 \oplus T_3{}^b \oplus T_3{}^3 \\ 3 \otimes \bar{3} \;&=\; 2 \otimes \bar{2} + 2 + \bar{2} + 1 \\ &=\; 3 + 1 + 2 + \bar{2} + 1 \end{aligned}$$

Note 1. $\mathrm{sl}(2,\mathbb{R})$ is special since 2 and $\bar{2}$ are equivalent! (From $\mathrm{sl}(2,\mathbb{R}) \cong \mathrm{sp}(2,\mathbb{R})$ with has $\varepsilon_{\mathrm{ab}}$ as inv tensor.) The 2 is called pseudo-real.

Can we find general methods to deal with representations?

Go back to $\mathrm{sl}(2,\mathbb{R})$:

Possible highest weight representations: $\Lambda = 0, 1, 2, ...,$ $\Rightarrow \dim = \Lambda + 1$, with $\Lambda = N \in \mathbb{Z} (= 2j \,\mathrm{in\,QM})$.

Algebra:

$$\begin{aligned}
[E_+, E_-] &= H \\
[H, E_\pm] &= \pm 2E_\pm
\end{aligned}$$

The 2 above is the Cartan matrix $A = 2$! So we have just one one-dimensional root α!

$$\begin{aligned}
\alpha &= 2 \\
A &= 2 \\
\text{but}\, A &= \alpha\cdot\alpha \qquad \big(\text{since}\,\alpha^{(i)}\cdot\alpha^{(j)} = A^{ij}\big)
\end{aligned}$$

But $2\times 2 = 4$ which is wrong, we need to go to the orthonormal basis! I.e. the metric must be $G^{-1} = \frac{1}{2}$, so that we have

$$\alpha\cdot\alpha = \frac{1}{2}2\times 2$$

Because we have that $G = \mathrm{TrH}H = 2$.

To get to the orthonormal basis and draw things:

$$\begin{aligned}
\tilde{\alpha} &= \frac{1}{\sqrt{2}}\alpha \\
\tilde{H} &= \frac{1}{\sqrt{2}}H \\
&\Rightarrow \\
\tilde{\alpha}\cdot\tilde{\alpha} &= 2 \qquad \text{with}\,\tilde{G} = 1
\end{aligned}$$

- Root space

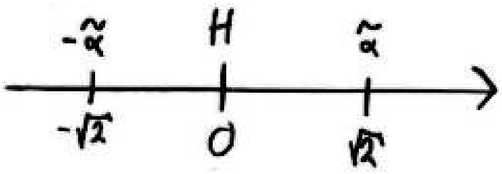

Figure 5.

- Root lattice

$$\{v|m\tilde{\alpha}, m\in\mathbb{Z}\}$$

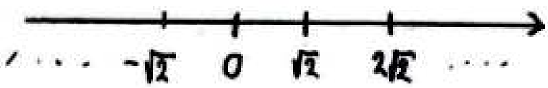

Figure 6.

- Weight lattice, dual of the root lattice

  Spanned by a vector $\tilde{\Lambda}$ s.t. $\tilde{\Lambda} \cdot \tilde{\alpha} = 1$. (Where the product is just the natural product between vectors and covectors).

$$\begin{aligned} \tilde{\alpha} &= \sqrt{2} \\ \tilde{\Lambda} &= \frac{1}{\sqrt{2}} \end{aligned}$$

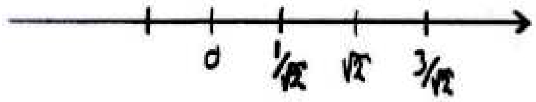

Figure 7.

- Weight spaces.

  Ex $\Lambda = 5 \Longrightarrow d = 6$.

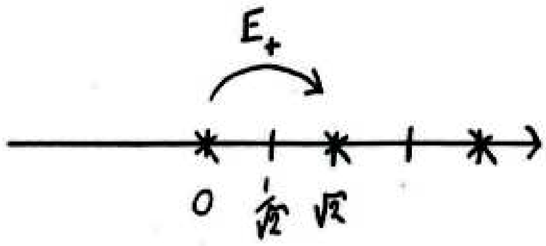

Figure 8.

Note 2. Here we view the root vectors as states in the adjoint representation so that $E_\pm$ can act on them. The generators H, E_+, E_- span a 3-dimensional vector space which is a representation of $\mathrm{sl}(2, \mathbb{R})$ with HWS=E_+ and LWS=E_-. The algebra of $\mathrm{sl}(2, \mathbb{R})$ acts on this vectorspace by the commutator. This action is usually denoted by $\mathrm{ad}_x(y) \equiv [x, y]$.

$$\begin{aligned} \mathrm{ad}_{E_+}(E_-) &= [E_+, E_-] \\ &= H \\ \mathrm{ad}_{E_+}(H) &= [E_+, H] \\ &= 2E_+ \\ \mathrm{ad}_{E_+}(E_+) &= [E_+, E_+] \\ &= 0 \end{aligned}$$

Thus E_+ steps through the three states in the representation and E_+ itself is the highest weight state as promised.

A_2

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
\alpha^{(1)} &= \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\alpha^{(2)} &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}
\end{aligned}
$$

The weight lattice is the dual lattice spanned by $\Lambda_{(1)}$ and $\Lambda_{(2)}$ s.t.

$$
\begin{aligned}
\Lambda_{(i)} \cdot \alpha^{(j)} &= \delta_i{}^j \\
&\Rightarrow \\
\Lambda_{(1)} &= \frac{1}{\sqrt{2}}\left(1, \frac{1}{\sqrt{3}}\right) \\
\Lambda_{(2)} &= \frac{1}{\sqrt{2}}\left(0, \frac{2}{\sqrt{3}}\right) \\
|\Lambda_{(i)}|^2 &= \frac{2}{3}
\end{aligned}
$$

All of the above is in the orthonormal basis.

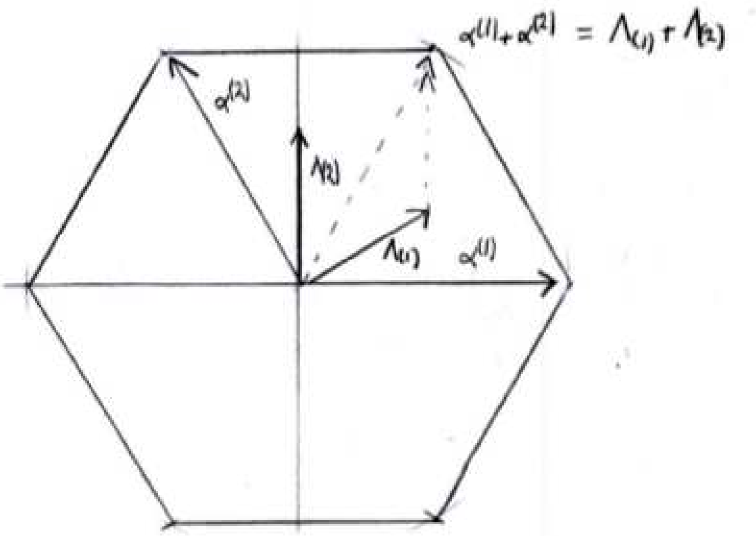

Figure 9.

Note

$$
\frac{\text{volume (root)}}{\text{volume (weight)}} = 3 = \text{Number of conjugacy classes}
$$

Definition 3.

Root lattice $= \{m_1\alpha^{(1)} + m_2\alpha^{(2)}, m_1, m_2 \in \mathbb{Z}\}$

Weight lattice $= \{n^1\tilde{\Lambda}_{(1)} + n^2\tilde{\Lambda}_{(2)}, n^1, n^2 \in \mathbb{Z}\}$

Definition 4.

HWS

$$\Lambda \;=\; \sum n^i \Lambda_{(i)} \qquad \text{with}\, n^i \geqslant 0$$

$s.t.$

$$\begin{aligned} E_+^i v_\Lambda &= 0 \\ H v_\Lambda &= \Lambda v_\Lambda \end{aligned}$$