2012–03–14

Lecturer: Bengt E W Nilsson

Recall: $\mathfrak{su}(2) \sim \mathfrak{sl}(2,\mathbb{R}) \sim \mathfrak{sl}(2,\mathbb{C})$ as complex vector spaces. The representation theory is independent of which one we mean, at this point. Unitarity of representations *will* depend on the choice of *real form*.

We will consider

$$L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} [L_+, L_-] = L_0 \\ [L_0, L_\pm] = \pm 2 L_\pm \end{cases}$$

$\Rightarrow$ Representations (i.e. modules) in general are constructed from highest-weight states (HWS) v_Λ, where Λ is the highest weight. They are such that

$$L_+ v_\Lambda = 0, \quad L_0 v_\Lambda = \Lambda\, v_\Lambda.$$

Stepping up gives zero. Step down with L_-:

$$v_\Lambda \to L_- v_\Lambda = v_{\Lambda-2}.$$

Check:

$$L_0(L_- v_\Lambda) = \underbrace{[L_0, L_-]}_{=-2L_-} v_\Lambda + L_- \underbrace{L_0 v_\Lambda}_{=\Lambda v_\Lambda} = (\Lambda - 2)(L_- v_\Lambda)$$

$$\Rightarrow L_- v_\Lambda = v_{\Lambda-2}$$

(There is a proportionality constant here, that we set to unity.) This repeats:

$$(L_-)^n\, v_\Lambda \equiv v_{\Lambda-2n}$$

and this stops after som N steps, i.e.

$$(L_-)^N v_\Lambda = v_{\Lambda-2N}$$

and

$$L_- v_{\Lambda-2N} = 0.$$

This means that you get the picture of a line.

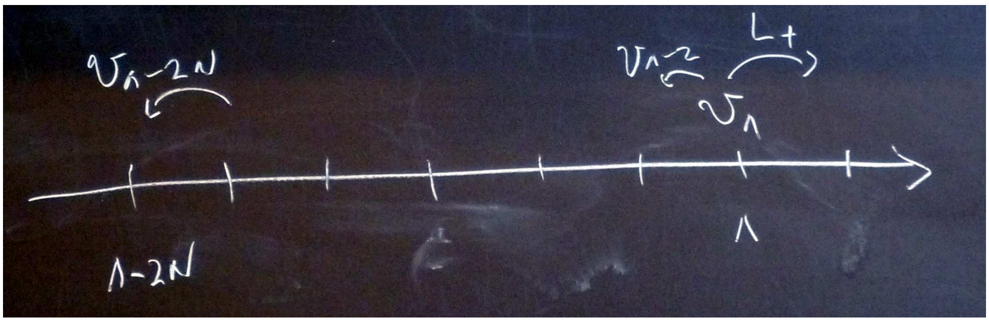

Figure 1.

$d = N + 1$ states in the representation (this is called the dimension).

Stepping up:

$$L_+ v_{\Lambda - 2n} \equiv r_n v_{\Lambda - 2n + 2}$$

$$\Rightarrow r_n = n(\Lambda - n + 1)$$

By computing

$$0 = L_+ L_- v_{\Lambda - 2N}$$

$$\Rightarrow N = \Lambda \text{ or } -1, \quad \text{so} \boxed{N = L}$$

It is symmetric about zero. $v_{\Lambda - 2N} = v_{-\Lambda}$.

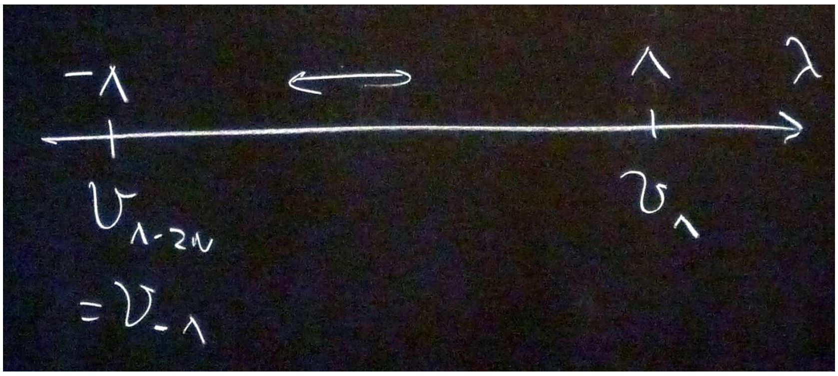

Figure 2. Symmetric about zero.

Comments:

$N = 2, d = 3.$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(new notation for L_0). This is the Cartan element (of the Cartan subalgebra).

Stepping-down operator:

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

It is often called E_-.

$$F \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[H, F] = -2\, F$$

Stepping-up operator:

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is often called E_+.

In $\mathfrak{sl}(3,\mathbb{R})$ we need three upper-triangular matrices! What about

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_+^2$$

in $\mathfrak{sl}(2,\mathbb{R})$. So, is $(E_+)^2$ in $\mathfrak{sl}(2,\mathbb{R})$?

No, since only commutators are new elements in a Lie algebra, and

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is *not* obtainable like that in $\mathfrak{sl}(2,\mathbb{R})$. (But it is, of course, in $\mathfrak{sl}(3,\mathbb{R})$.)

$$E_+^1 \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_+^2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_+^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\mathfrak{sl}(2,\mathbb{R}) \equiv A_1$ (Cartan's classification)

$$\begin{aligned} A_1\colon\ & [E_+, E_-] = H, \\ & [H, E\pm] = \pm 2\, E_\pm. \end{aligned}$$

The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$, or A_2

Try to combine two $\mathfrak{sl}(2,\mathbb{R})$'s into a more complicated algebra, and finally to get $\mathfrak{sl}(3,\mathbb{R})$.

First $\mathfrak{sl}(2,\mathbb{R})$:

$$\begin{aligned} &[E_+^1, E_-^1] = H^1 \\ &[H^1, E_\pm^1] = \pm 2\, E_\pm^1 \end{aligned}$$

To get a second quantum number (which is just an index enumerating the states and eigenvectors), we need a new element that commutes with H^1, which we call H^2.

$$[H^1, H^2] = 0.$$

This we call rank 2. The rank is the number of Cartan subalgebras.

If H^2 commutes also with $E_\pm^1$ then nothing interesting happens.

Example:

$$H^2 = \sum_i (L_i)^2 = C_2 \quad \Rightarrow \quad L_i^2|\lambda\rangle = j(j+1)|\lambda\rangle \text{ where } \Lambda = 2\,j$$

To really get something new, H^2 must be part of a second $\mathfrak{sl}(2,\mathbb{R})$:

Second $\mathfrak{sl}(2,\mathbb{R})$:

$$[H^2, E_\pm^2] = \pm 2\, E_\pm^2$$

$$[E_+^2, E_-^2] = H^2$$

Now, if all generators in the first $\mathfrak{sl}(2,\mathbb{R})$ commutes with all in the second one, the new algebra is "trivial":

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \quad \dim = 6.$$

All states are labelled by two numbers, the eigenvalues of H^1 and H^2.

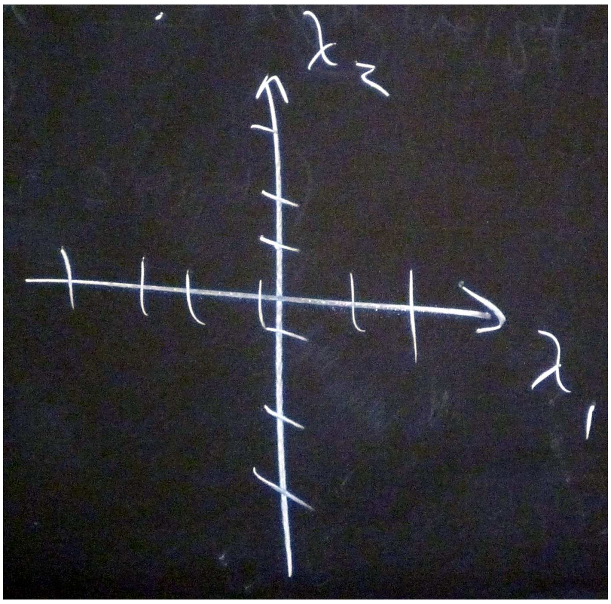

Figure 3.

So, to get an even larger algebra (like $\mathfrak{sl}(3,\mathbb{R})$, which is $\dim = 8$) the two $\mathfrak{sl}(2,\mathbb{R})$'s cannot commute:

$$\left[H^i, E^j_{\pm}\right] = \pm A^{ji} E^j_{\pm}.$$

Note the order of the indices here. Note also that $A^{11} = A^{22} = 2$, from the already established commutation relations.

To get a mixing between the $\mathfrak{sl}(2,\mathbb{R})$'s we need A^{21} and/or A^{12} to be non-zero.

Consider the other possible commutators, between the E's:

$$\begin{array}{l} [E^1_{\pm}, E^2_{\pm}] \equiv E^{\theta}_{\pm} \quad \text{(this is just notation, at this point)} \\ [E^1_{\pm}, E^2_{\mp}] \end{array}$$

We now would like to relate their eigenvalues of H^i to A^{ji} (if possible).

1)

$$[H^i, [E^1_{\pm}, E^2_{\pm}]] = [H^i, E^{\theta}_{\pm}]$$

$$[H^i, [E^1_{\pm}, E^2_{\pm}]] = [\text{Jacobi}] = -[E^1_{\pm}, [E^2_{\pm}, H^i]] - [E^2_{\pm}, [H^i, E^1_{\pm}]] =$$

$$= -[E^1_{\pm}, \mp A^{2i} E^2_{\pm}] - [E^2_{\pm}, [H^i, \pm A^{1i} E^1_{\pm}]] =$$

$$= \pm (A^{1i} + A^{2i}) \, [E^1_{\pm}, E^2_{\pm}]$$

also

$$[H^i, [E^1_\pm, E^2_\mp]] = \pm(A^{1i} - A^{2i})[E^1_\pm, E^2_\mp]$$

Hence: both $E^\theta_\pm$ and $[E^1_\pm, E^2_\mp]$ are possible new elements in the bigger Lie algebra. To find the *smallest* extension of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, we will try to keep $E^\theta_\pm$ and set $[E^1_\pm, E^2_\mp]$ to zero.

So: We assume

(1) $[E^1_\pm, E^2_\mp] = 0$ and

(2) $[E_\pm, E^\theta_\pm] = 0$ and

(3) $[E^i_\mp, E^\theta_\pm] \neq 0$ by Jacobi.

Now compute

$$\left[E^1_+, \underbrace{[E^1_-, E^2_+]}_{=0\text{ by (1)}}\right] - \left[E^1_-, \underbrace{[E^1_+, E^2_+]}_{E^\theta_+}\right] = [\text{Jacobi}] = -\left[E^2_+, \underbrace{[E^1_+, E^1_-]}_{H^1}\right]$$

$$= +[H^1, E^2_+] = A^{21}E^2_+$$

We can connect the existence of the opreator E^θ_+ with A^{21} being nonzero.

$$\text{Thus}\quad \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}) \Rightarrow A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Can we get A^{21}? Yes.

$$0 = \left[E^1_-, \underbrace{[E^1_+, E^\theta_+]}_{=0\text{ by (2)}}\right] = [\text{Jacobi}] = \ldots = -(2\,A^{21} + A^{11})E^\theta_+$$

$$\Rightarrow A^{21} = -1$$

also $A^{12} = -1$.

Thus we know the *Cartan matrix* for A_1.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

To summarize $\mathfrak{sl}(3,\mathbb{R})$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad\Leftrightarrow\quad \begin{cases} [H^1, H^2] = 0 \\ \left[H^i, E^j_\pm\right] = \pm A^{ji}E^j_+ \\ \text{There are altogether 6 step operators: } E^i_\pm, E^\theta_\pm \end{cases}$$

$\dim(A_2) = 8$.

Note:

$$[H^i, E^\theta_\pm] = \pm(A^{1i} + A^{2i})E^\theta_+$$

Note:

$$[E^1_\pm, E^2_\pm] = \pm E^\theta_\pm, \quad [E^\theta_+, E^\theta_-] = H^1 + H^2$$

This means that there's a third $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{sl}(3,\mathbb{R})$. Exercise!

The third $\mathfrak{sl}(2,\mathbb{R})$:

$$E^3_\pm \equiv E^\theta, \quad H^3 = -(H^1 + H^2).$$

$$\Rightarrow H^1 + H^2 + H^3 = 0$$

and all three $\mathfrak{sl}(2,\mathbb{R})$'s are completely equivalent inside $\mathfrak{sl}(3,\mathbb{R})$.

In physics the $\mathfrak{sl}(2,\mathbb{R})$'s are called (weak) isospin U and V.

Question: How can we make sense of all these $\mathfrak{sl}(3,\mathbb{R})$ commutators?

Let the Cartan subalgebra $(H^1, H^2) = H^i$ span a vector space

$$h = h_i H^i$$

(like writing $\boldsymbol{r} = x_i \boldsymbol{e}^i$).

Then $\left[H^i,\, E^j_\pm\right] = \pm\left(\alpha^{(j)}\right)^i E^j_+$ where $\alpha^{(j)}$ is a vector of eigenvalues for the operator $E^j_\pm$, i.e. $A^{ji} = \left(\alpha^{(j)}\right)^i$ is the ith component of the eigenvalue vector $\alpha^{(j)}$.

$$A^{ji} = \begin{array}{cl} \left(\begin{array}{cc} A^{11} & A^{12} \\ A^{12} & A^{22} \end{array}\right) & \begin{array}{l} \longleftarrow \left(\alpha^{(1)}\right)^i \\ \longleftarrow \left(\alpha^{(2)}\right)^i \end{array} \end{array}$$

so for E^1_+ we get $\alpha^{(1)} = (A^{11}, A^{12}) = (2, -1)$.

For E^2_+ we get $\alpha^{(2)} = (A^{21}, A^{22}) = (-1, 2)$.

For $E^\theta_\pm$ we get $\theta = \left(\alpha^{(1)} + \alpha^{(2)}\right) = (1, 1)$.

Also E^i_- give $-\alpha^{(i)}$ and E^θ_- gives $-\theta$.

Standard is to draw these in a root diagram.

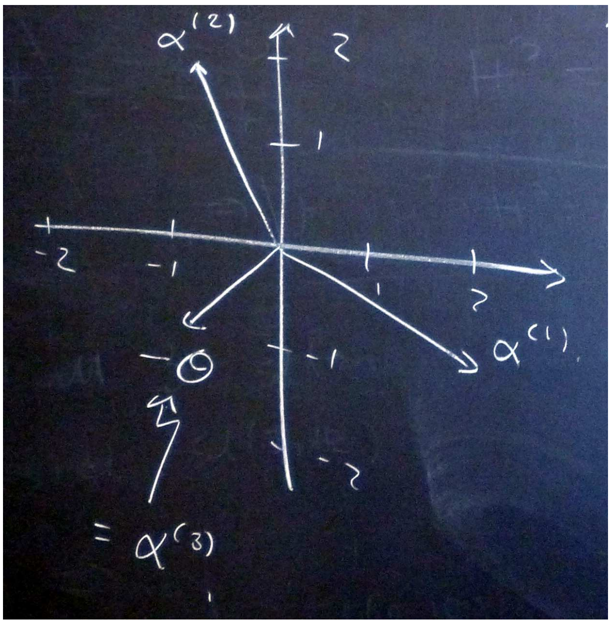

Figure 4. The symmetry between the 3 $\mathfrak{sl}(2,\mathbb{R})$'s not present.

What is wrong? Answer: We have assumed something in this picture that is not natural (canonical).

We have introduced the metric $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in this picture. But there is a canonical metric:

Recall $h = h_i H^1$ with

$$H^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad H^3 = \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$E_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \dots$$

Lie algebras of $\mathfrak{sl}(3, \mathbb{R})$.

Metric: $G^{ij} = \mathrm{tr}(H^i, H^j) = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) = A^{ji}$ again.

$$\Rightarrow h \cdot h' = h_i \, G^{ij} h_j$$

$$\alpha^{(i)} \cdot \alpha^{(j)} = \left(\alpha^{(i)}\right)^k \left(\alpha^{(j)}\right)^l G_{kl}$$

where G_{kl} is by definition the matrix inverse of G^{ij}.

$$G_{ij} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Compute, with scalar product given by G:

$$\alpha^{(i)} \cdot \alpha^{(j)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A$$

To redraw the diagram, we have to diagonalize the metric G^{ij}: Define

$$\tilde{H}^i = \left(\frac{1}{\sqrt{2}} H^1, \frac{1}{\sqrt{6}} (H^1 + 2\, H^2) \right) = M^i{}_j \, H^j$$

$$\tilde{G}^{ij} = \mathrm{tr}\big(\tilde{H}^i \tilde{H}^j\big) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \left[\tilde{H}^i, E_\pm^i\right] = M^i{}_j \big(\alpha^{(1)}\big)^j \, E_\pm^1$$

$$\begin{cases} \tilde{\alpha}^{(1)} = \sqrt{2} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \\ \tilde{\alpha}^{(2)} = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} -1 \\ \sqrt{3} \end{smallmatrix}\right) \\ \tilde{\alpha}^{(3)} = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 \\ \sqrt{3} \end{smallmatrix}\right) \end{cases}$$

With the scalar product given by δ^{ij}:

$$\tilde{\alpha}^{(i)} \cdot \tilde{\alpha}^{(j)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A$$

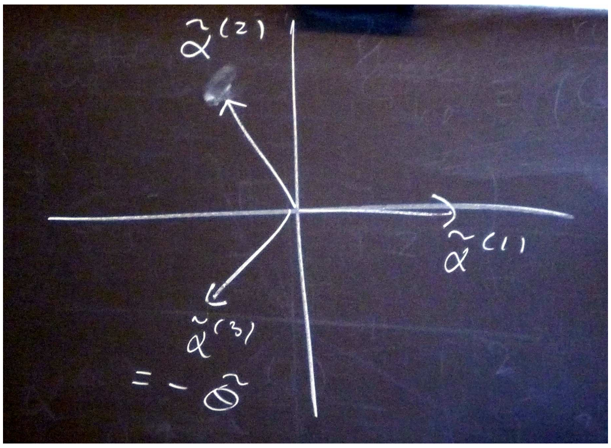

Figure 5.

Note $\tilde{\alpha}^{(1)} + \tilde{\alpha}^{(2)} + \tilde{\alpha}^{(3)} = 0$.

Comments:

$\mathfrak{sl}(3,\mathbb{R})$ contains $H^1, H^2, E^i_{\pm}, E^{\theta}_{\pm}$

There is a “metric” on the whole vector space. Killing form κ^{ab}:

$$\kappa^{ab} = \begin{array}{cl} \left(\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{array}\right) & \begin{array}{l} H^1 \\ H^2 \\ E^i_+ \\ E_-{}^i \end{array} \end{array}$$

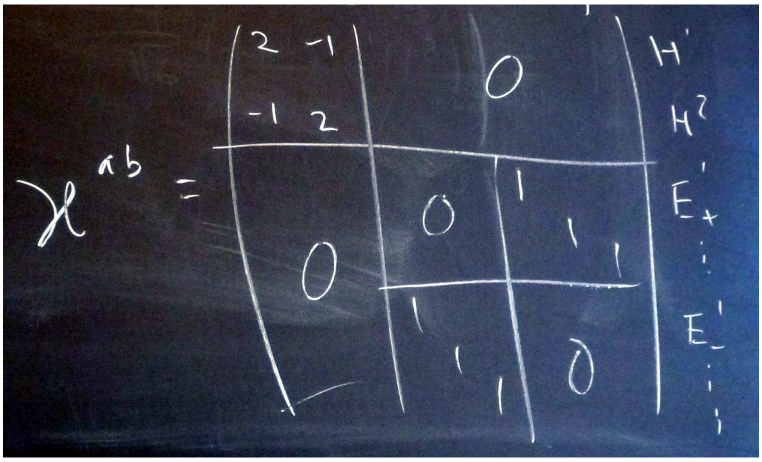

Figure 6.

Split form.

Diagonalize κ in the non-Cartan part.

$$\kappa = \left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & \mathbb{1} & 0 \\ 0 & 0 & -\mathbb{1} \end{array}\right)$$

In this basis we have $H^1, H^2, E^i_+ + E^i_-, E^i_+ - E^i_-$ for $i = 1, 2, 3$.

Note

$$E_+^1 - E_-^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{e}^{\alpha(E_+^1 - E_-^1)} = \cos\alpha + (\ldots)\sin\alpha \quad \Rightarrow \text{compact.}$$

while all other $\mathrm{e}^{\cdots}$ give $\cosh + \sinh \Rightarrow$ non-compact.

$E_+ - E_-$ generate the maximal compact subgroup of $\mathfrak{sl}(3, \mathbb{R})$ and the coset $\mathfrak{sl}(3, \mathbb{R})/\mathfrak{so}(3) \approx$ 5-dimensional hyperbolic.

Compare $\mathfrak{sl}(2, \mathbb{R})/\mathfrak{so}(2)$.

Cartan matrices:

Rank 2:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det A = 3, \quad A_2 \text{ (dim 8)}$$

The off-diagonal elements we could change:

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad \det A = 2, \quad B_2 \text{ (dim 10)}$$

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad \det A = 1, \quad G_2 \text{ (dim 14)}$$

The diagonal is always $(2, 2)$, the off-diagonal elements always negative, the determinant always possible.