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Recall: $\mathrm{SL}(2,\mathbb{Z})$ with elements

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d\in\mathbb{Z}, \quad \det g=1.$$

This leads to Bézout's identity $p_1\,q_2-p_2\,q_1=d$ where $d=\gcd(p_1,p_2)$. Then you can use Euclid's algorithm to get a unique answer for (q_1,q_2), that we may call $(\bar{q}_1,\bar{q}_2)$. There are other answers, that you can write by shifting the $(\bar{q}_1,\bar{q}_2)$ that Euclid's algorithm gives you.

$$(q_1,q_2)=(\bar{q}_1,\bar{q}_2)+\frac{n}{d}(p_1,p_2) \quad \text{for} \quad n\in\mathbb{Z}.$$

Useful relations for $\mathrm{e}^A\in G, A\in\mathrm{Lie}(G)$:

$$(\mathrm{e}^A)^{\mathrm{T}}=\mathrm{e}^{(A^{\mathrm{T}})}$$

$$(\mathrm{e}^A)^{\dagger}=\mathrm{e}^{(A^{\dagger})}$$

$$\mathrm{e}^A\,\mathrm{e}^B=\mathrm{e}^{A+B} \text{ if and only if } [A,B]=0, \text{ by CBH}$$

$$\Rightarrow(\mathrm{e}^A)^{-1}=\mathrm{e}^{-A}$$

$$\det\mathrm{e}^A=\mathrm{e}^{\mathrm{tr}\,A}$$

If $R\in\mathrm{SO}(N)$ then $R^{\mathrm{T}}R=\mathbb{1}$, implying that for $R=\mathrm{e}^A$ we get $A^{\mathrm{T}}=-A$. Antisymetric matrices in the Lie algebra.

If $U\in\mathrm{U}(N)$ and $U=\mathrm{e}^A$ we get $A^{\dagger}=-A$. Antihermitian matrices in the Lie algebra.

$g\in\mathrm{SL}(N,\mathbb{F})$ and $g=\mathrm{e}^A$ then $\mathrm{tr}\,A=0$.

Example. SO(3). A: 3×3 antisyummetric real matrices. One basis of $\mathrm{Lie}(\mathrm{SO}(3))=\mathfrak{so}(3)$ is

$$T^1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T^2=\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^3=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}: \quad (T^i)^{jk}=\varepsilon^{ijk}.$$

$$\Rightarrow[T^i,T^j]=\underbrace{-\varepsilon^{ijk}}_{f^{ijk}}T^k$$

The f^{ijk} are called structure constants.

Note: $[T,T]=f^{abc}T$ is valid for antihermitian generators. But often in physics we have hermitian operators J, in which case we will need an imaginary unit here: $[T,T]=\mathrm{i}\,f^{abc}J$.

Adjoint representation (or regular) for any $\mathrm{Lie}(G)$

This is formed by the $f^{ab}{}_c$'s:

$$(T^a)^b{}_c=-f^{ab}{}_c$$

$$\Rightarrow(T^a)^d{}_e(T^b)^e{}_f-(T^b)^d{}_e(T^a)^e{}_f=f^{ab}{}_c(T^c)^d{}_f$$

This is also seen to be the Jacobi identity.

Exercise: Check this.

Question: How can we analyze Lie algebras to get information which is independent of any chosen basis (in the Lie algebra itself or in the modules)? We will do this in Chapter 6.

Examples from physics: two fundamental examples.

Example 1: The Virasoro algebra (derived in the String Theory course).

Consider a two-dimensional space with coordinate $z \in \mathbb{C}$. Then the *vector fields*

$$V_n = z^{-n+1}\partial_z, \quad n \in \mathbb{Z}$$

$$\Rightarrow \quad [V_m, V_n] = z^{-m+1}\partial_z z^{-n+1}\partial_z - (m \leftrightarrow n) = \ldots = (m-n)V_{m+n}$$

Here $m-n$ are the structure constants: this is now an infinite-dimensional Lie algebra. This is known as the Witt algebra, satisfying the Jacobi identity.

Note: "All sets" of vector fields give Lie algebras (for instance Killing vectors give the isometry group).

Note: If we set $|z| = 1$ the $z \in S^1$ and the Witt algebra becomes $\mathrm{Vect}(S^1)$ and the Lie algebra of the group $\mathrm{Diff}(S^1)$.

In string theory, coindensed matter theory (phase transitions etc) this Witt algebra has an extra term ($V_n \to L_n$):

$$\underbrace{[L_m, L_n] = (m-n)L_{m+n}}_{\text{Witt algebra part}} + \underbrace{\frac{c}{12}m(m^2-1)\,\delta_{m+n,0}}_{\substack{\text{Central extension:}\\ \text{This is allowed by the}\\ \text{Jacobi identity}}}$$

This is the Virasoro algebra. In the central extension: c is an operator that commutes with L_m (that's the meaning of *central* here: elements that commute with everything in the algebra).

In physics c ius called the *conformal anomaly*, and it comes from *normal ordering* effects in quantum field theory.

Exercise: Show that the central extension is allowed by the Jacobi identity.

Note: In Quantum Field Theory one is interested in representations (e.g. of the Virasoro algebra) with the properties

1) Unitary (conservation of probability in Quantum Mechanics), and

2) Highest-weight representations (this has to do with the energy being bounded from below).

Then for the Virasoro algebra such representations exist only if $c > 0$. (The Witt algebra fails here.)

Example 2: Classical Poisson algebra:

Hamilton says:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad H = H(p,q)$$

For $f(p,q)$, a function in phase space:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(p,q) = \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} \overset{\text{(Hamilton)}}{=\!=\!=\!=} \{H, f\}_{\mathrm{PB}} \text{ where } \{A, B\}_{\mathrm{PB}} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$$

This is antisymmetric, and satisfies the Jacobi identity. (Exercise: check!)

This means that the Poisson bracket is a Lie product.

Chapter 5: Representation theory: Introduction

Recall the angular momentum algebra from Quantum Mechanics:

$$\boldsymbol{L} = \boldsymbol{q} \times \boldsymbol{p}, \quad L^i = \varepsilon^{ijk} q^j p^k \quad \text{with } [q^i, p^j] = \mathrm{i}\,\delta^{ij}$$

$$\Rightarrow [L^i, L^j] = \mathrm{i}\varepsilon^{ijk} L^k$$

This is the $\mathfrak{so}(3)$ Lie algebra. So this is (at least) the third realization of this algebra. Before 2×2 matrices and 3×3 matrices. $\mathrm{Lie}(\mathrm{SU}(2)) \approx \mathrm{Lie}(\mathrm{SO}(3))$.

$$\mathrm{SU}(2): \quad T^a = \frac{1}{2}\sigma^a$$

$$\mathrm{SO}(3): \quad (T^i)^{jk} = -\varepsilon^{ijk}$$

Question: Can we classify all possible representations of a Lie algebra? Yes, for finite dimensional ones (done below).

To find all finite-dimensional (matrix representation) we form

$$\begin{cases} L_{\pm} = L_1 \pm \mathrm{i}\, L_2 \\ L_0 = 2 L_3 \end{cases}$$

Structure constants:

$$\begin{cases} f_{0\pm}{}^{\pm} = \pm 2 \\ f_{+-}{}^{0} = 1 \end{cases}$$

$\mathfrak{sl}(2, \mathbb{R})$ Lie algebra.

$$L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\begin{cases} [L_0, L_{\pm}] = \pm 2\, L_{\pm}, \\ [L_+, L_-] = L_0. \end{cases}$$

We will analyze all Lie algebras related by complex numbers to this one at the same time. We will not distinguish $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{C})$. This means that unitarity must be checked afterwards.

Theorem: Non-compact Lie algebras (like $\mathfrak{sl}(2, \mathbb{R})$; $\mathfrak{su}(2)$ *is* compact) do not have any unitary finite-dimensional representations.

Then: Consider L_0. It has at least one non-zero eigenvalue (and eigenvector). Write this as

$$L_0 v_\lambda = \lambda\, v_\lambda$$

where v_λ is the eigenvector and λ is the eigenvalue, called the *weight*. The idea is to use the algebra to construct more eigenvectors. We have that $L_\pm v_\lambda$ will also be eigenvectors with eigenvalues $\lambda \pm 2$:

$$L_0(L_\pm v_0) = (\lambda \pm 2)(L_\pm v_\lambda)$$

Let us denote these new vectors $v_{\lambda \pm 2} \propto L_\pm v_\lambda$, where we will decide the proportionality constant later.

Check:

$$L_0(L_\pm v_0) = \underbrace{[L_0, L_\pm]}_{= \pm 2 L_\pm} v_\lambda + L_\pm \underbrace{L_0 v_\lambda}_{= \lambda v_\lambda} = (\lambda \pm 2)(L_\pm v_\lambda).$$

Please memorize this little calculation.

Example: If $T^a \sim \sigma^a$, 2×2 matrices,

$$L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then we have states $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$:

$$L_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is the hightest-weight state.

So in general there exists some highest weight Λ and a highest weight state v_Λ such that $L_+ v_\Lambda = 0$. We can also move down from v_Λ with L_- until it stops.

Figure 1.

We suppose that after N steps we reach the lowest weight state $v_{\Lambda - 2N}$.

$$(L_-)^N v_\Lambda = v_{\Lambda - 2N} \quad \text{and} \quad L_- v_{\Lambda - 2N} = 0,$$

which span a $d = N + 1$ dimensional space.

Here we defined the exact relation between v_Λ and $L_- v_\Lambda$: i.e. we define with coefficient one $L_- v_\lambda = v_{\lambda - 2}$, etc.

Then if we step up we get numbers r_n:

$$L_+ v_{\Lambda - 2} = L_+ L_- v_\Lambda = \underbrace{[L_+, L_-]}_{= L_0} v_\lambda + L_- \underbrace{L_+ v_\Lambda}_{= 0} = L_0 v_0 = \Lambda v_\Lambda.$$

Define r_n:

$$L_+ v_{\Lambda - 2n} \equiv r_n v_{\Lambda - 2n + 2}$$

Repeating this we get the relation $r_n = r_{n-1} + \Lambda - 2\,n + 2$, where we define $n = 0, 1, 2$ by keeping $r_0 = 0$. We get

$$r_n = n\,(\Lambda - n + 1)$$

At this point we have two unrelated numbers N and Λ. They can be related by

$$0 = L_+ L_- v_{\Lambda - 2N} = (L_- L_+ + L_0) v_{\Lambda - 2N} = (r_N + \Lambda - 2\,N) \underbrace{v_{\Lambda - 2N}}_{\substack{\neq 0 \\ \text{LWS}}}$$

$$r_N + \Lambda - 2N = 0$$

$$\Lambda = 2\,N - N(\Lambda - N + 1)$$

$$N = -\frac{1 - \Lambda}{2} \pm \sqrt{\Lambda + \left(\frac{1 - \Lambda}{2}\right)^2}$$

This becomes Λ (or -1). So $N = \Lambda$.

Thus the highest weight state is $v_\Lambda = v_N$ and the lowest weight state is $v_{\Lambda-2N} = v_{-\Lambda}$. We jump with steps of two all the time.

Quantum Mechanics: $\Lambda = 2\,j$ with $j = 0, \frac{1}{2}, 1, \ldots$

States $|j, m\rangle$. We have $L_+|j, j\rangle = 0$ and $L_3 = |j, m\rangle = m|j, m\rangle$.

We also have (Casimir) $\boldsymbol{L}^2|j, m\rangle = j(j+1)|j, m\rangle$. Is the operator $\boldsymbol{L}^2$ in the Lie algebra? No.

Note: $[\boldsymbol{L}^2, L_\pm] = [\boldsymbol{L}^2, L_0] = 0$.

What about Schur's lemma? If we have an operator that commutes with the entire Lie algebra, why isn't it trivial?

In Quantum Mechanics one often defines $L_\pm v$ with $\sqrt{r_n}$.