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Start from Lie group, find coordinates on the group — group expressed in terms of a manifold.

*Recall:* Example: $\mathrm{SU}(2) \approx S^3$.

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \Rightarrow \quad \det g = |\alpha|^2 + |\beta|^2 = 1$$

We considered two different sets of coordinates on $S^3$:

1) Euler angles. These were bad at the identity.

2) $\theta_1, \theta_2, \phi$ was a good coordinate system.

$\Rightarrow$ Changing coordinates from one good set to another $\Rightarrow$ generators belong to a *vector space*.

Example. Compare *generators* of SU(2) and SL$(2,\mathbb{R})$.

$$\begin{aligned} &\mathrm{SU}(2): && J^i = \frac{1}{2}\sigma^i, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\mathrm{SL}(2,\mathbb{R}): && T^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Note that $\sigma^2$ is complex, but the $T^i$ are real. But the coordinates are real. If you allow complex coefficients, there would be no difference between these two.

Comments:

1) Generators belong to vector spaces. (Change of (good) coordinates on the group $\Rightarrow$ linear combinations of the generators.

2) Lie algebra Lie$(G)$ is a vector space with a product: the Lie product. We will just denote it with a bracket $[\cdot,\cdot]$ just like a commutator — but it doesn't mean it *has* to be a commutator.

3) Even if the generators are complex the coordinates on $G$ are real. We will soon discuss the relevance when we allow for complex linear combination in the Lie algebra (complexification: $\mathfrak{sl}(2,\mathbb{R}) \sim \mathfrak{su}(2) \sim \mathfrak{sl}(2,\mathbb{C})$).

Maurer–Cartan: $g \in G$: $\omega = g^{-1}\mathrm{d}g \in \mathrm{Lie}(G)$. Since

$$\mathrm{d} = \mathrm{d}x^a \frac{\partial}{\partial x^a}$$

At the identity:

$$\omega = (g^{-1}\mathrm{d}x^a \partial_a g)|_{\mathbb{1}} = \mathrm{d}x^a (\partial_a g)|_{\mathbb{1}} = \mathrm{d}x^a T^a$$

but the definition $\omega = g^{-1}\mathrm{d}g$ works at any point on $G$.

This is left-invariant: $g \to g_0 g \Rightarrow \omega \to \omega$ where $g_0$ is a constant matrix in $G$.

Maurer–Cartan equation: view d and view $\omega$ as 1-forms:

$$\mathrm{d}\omega = \mathrm{d}(g^{-1}\mathrm{d}g) = (\mathrm{d}g^{-1})\mathrm{d}g + g^{-1}\mathrm{d}^2 g = (\mathrm{d}g^{-1})\mathrm{d}g$$

$\mathrm{d}^2 = 0$ since $\mathrm{d}x^a \partial_\alpha \wedge \mathrm{d}x^b \partial_b$, where the wedge product is anti-symmetric. Forms are multiplied in an antisymmetric way. $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$. $\mathrm{d}x^a \wedge \mathrm{d}x^b = -\mathrm{d}x^b \wedge \mathrm{d}x^a$.

$$\mathrm{d}^2 = \mathrm{d}x^a \wedge \mathrm{d}x^b\, \partial_a \partial_b = -\mathrm{d}x^b \wedge \mathrm{d}x^a\, \partial_a \partial_b = -\mathrm{d}^2 \quad \Rightarrow \quad \mathrm{d}^2 = 0.$$

Then

$$\mathrm{d}\omega = (\mathrm{d}g^{-1})(\mathrm{d}g)$$

$$\mathrm{d}(g^{-1}g) = 0 \quad \Rightarrow \quad (\mathrm{d}g^{-1})g = -g^{-1}\mathrm{d}g$$

$$\mathrm{d}\omega = (\mathrm{d}g^{-1})(\mathrm{d}g) = (\mathrm{d}g^{-1})g\,g^{-1}(\mathrm{d}g) = -(g^{-1}\mathrm{d}g)(g^{-1}\mathrm{d}g)$$

$$\mathrm{d}\omega + \omega \wedge \omega = 0$$

**Gauge theory**

Let the field be written as $\phi$ which is a "vector" under some group $G$.

Example: In the Standard Model $\begin{pmatrix} e \\ \nu \end{pmatrix}_{\mathrm{L}}$ sit in a two-dimensional ("vector") representation of SU(2). The same for SU(3), etc.

Then $\bar{\phi}\phi$ is invariant, where $\bar{\phi} = \phi^\dagger$, which means that $\phi$ is covariant:

$$\phi \xrightarrow{g} \phi' = g\,\phi \quad \Rightarrow \quad \bar{\phi} \to \bar{\phi}\,' = \bar{\phi}\,g^\dagger \quad \Rightarrow \quad \bar{\phi}\phi = \bar{\phi}\,'\phi' \quad \text{since } g^\dagger g = \mathbb{1} \text{ in SU(2)}.$$

In SU(2):

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xrightarrow{g} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathrm{SU}(2)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

In physics we need to form invariants (in the Lagrangian for instance), from $\partial_\mu\phi$, but $\partial_\mu\phi$ is not covariant.

$$\partial_\mu\phi \xrightarrow{g} (\partial_\mu\phi)' = \partial_\mu(g\phi) \neq g\,(\partial_\mu\phi)$$

if $\partial_\mu g \neq 0$, i.e. for $g = g(x)$, i.e. $g$ is a *local symmetry* or *gauge symmetry*. NB: This means that the coordinates on $G$ are functions of the $x^\mu$: $\alpha^a = \alpha^a(x^\mu)$.

We can fix this up by forming a *covariant derivative* $\mathrm{D}_\mu = \partial_\mu + A_\mu$ where $A_\mu \in \mathrm{Lie}(G)$, and define a transformation rule for $A_\mu$. The rule for $A_\mu$ follows from demanding that $\mathrm{D}_\mu\phi$ is covariant:

$$\mathrm{D}_\mu\phi \xrightarrow{g} (\mathrm{D}_\mu\phi)' = g(\mathrm{D}_\mu\phi)$$

$$\Rightarrow \mathrm{D}'_\mu\phi' = g(\mathrm{D}_\mu g^{-1}\phi')$$

Drop $\phi' \Rightarrow$

$$\mathrm{D}'_\mu = g\,\mathrm{D}_\mu\,g^{-1}$$

In detail:

$$\partial_\mu + A'_\mu = g(\partial_\mu + A_\mu)g^{-1} = \underbrace{g\,\partial_\mu\,g^{-1}}_{=\partial_\mu + g(\partial_\mu g^{-1})} + g\,A_\mu\,g^{-1}$$

$$A'_\mu = (g\,\mathrm{D}_\mu g^{-1}) \in \mathrm{Lie}(G).$$

$$[\mathrm{D}_\mu, \mathrm{D}_\nu]\,\phi = \ldots = F_{\mu\nu}\,\phi$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$.

Exercise: Verify that $F_{\mu\nu} \xrightarrow{g} g\,F_{\mu\nu}\,g^{-1}$ when $A_\mu \to g\,\mathrm{D}_\mu\,g^{-1}$.

In form language

$$F = \mathrm{d}A + A \wedge A$$

(This is completely coordinate independent.)

Exercise: Use $F = \frac{1}{2}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu F_{\mu\nu}$ to check that this $F_{\mu\nu}$ is the one above.

Example: In two-dimensional spacetime:

$$\int_{S^2} F \text{ if } F \text{ in } \mathrm{U}(1).$$

This is related to the monopole number.

In three dimensions

$$\frac{1}{2}\int \left( A\,\mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right)$$

This is Chern–Simons. Exercise: Show that this is gauge invariant.

In four dimensions:

$$\int F \wedge F$$

gives you instanton numbers.

$$\int F \wedge \star F$$

Yang–Mills action.

Exercise: For U(1): $F = \mathrm{d}A$. Show that for any $G$, $F \wedge F = \mathrm{d}J$ and find $J$.

You also get General Relativity for free. On a curved manifold $\Rightarrow$ introduce tangent space (which you have to do, otherwise you can't describe spinors). Let $A$ be spin connection, and then $F$ will be the Riemann tensor (in a first order formulation, more later perhaps).

**Exponentiation of Lie algebras**

We have seen that linearization of a Lie group $G$ leads to its Lie algebra.

Definition. An *algebra* $\mathcal{A}$ is a vector space over a field $\mathbb{F}$ (reals, complex, quaternions, number fields) with a product $\times : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, that is *bilinear* and *distributive*. Bilinear menas that for $\alpha$, $\beta \in \mathbb{F}$ and $X, Y \in \mathcal{A}$:

$$(\alpha X) \times (\beta Y) = \alpha \beta X \times Y$$

Distributive means $\alpha(X + Y) = \alpha X + \beta Y$.

Definition. A *Lie algebra* is an algebra with a product called *Lie product* with the properties:

1) It is anti-symmetric: $[X, X] = 0$. $(X = y + z \Rightarrow [y + z, y + z] = 0 \Rightarrow [y, z] + [z, y] = 0)$.

2) It satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Note: If $x\,y$ is defined and hence $[x, y] = x\,y - y\,x$, then the Jacobi identity is trivial. If the $x\,y$ product is not defined, then the Jacobi identity is not trivial and is just part of the definition of a Lie algebra. It is needed to go backwards from the Lie algebra to the Lie group.

Can we reobtain the Lie group from the Lie algebra? One standard way is to repeat small steps in some Lie algebra direction $X\in\mathrm{Lie}(G)$ and infinite number of times. $X\in\mathrm{Lie}(G)$ means that $X$ is a linear combination of the generators of the algebra: $X=x^aT^a$.

$$f(X)=\lim_{n\to\infty}\left(\mathbb{1}+\frac{X}{n}\right)^n$$

*Note:* If $X$ is a number, then $\partial_X f=f\Rightarrow f=\mathrm{e}^X$.

So $f(X)=\mathrm{e}^X$.

*Note:* we can simply view this as

$$\mathrm{e}^X=\mathbb{1}+X+\frac{1}{2}X^2+\cdots$$

Is this useful? Can we compute $\mathrm{e}^X$ in general? Yes, but tricky!

EXAMPLE: $\mathrm{SL}(2,\mathbb{R})$ again:

$$X=x\,T^1+y\,T^2+z\,T^3=\begin{pmatrix} x & y\\ z & -x\end{pmatrix}$$

$$\Rightarrow\quad \mathrm{e}^X=\mathrm{e}^{xT^1+yT^2+zT^3}=\sum_{n=0}^{\infty}\frac{1}{n!}(X)^n=\begin{pmatrix} \cosh\theta+x\,\frac{\sinh\theta}{\theta} & y\,\frac{\sinh\theta}{\theta}\\ z\,\frac{\sinh\theta}{\theta} & \cosh\theta-x\,\frac{\sinh\theta}{\theta}\end{pmatrix}$$

Two ways:

1)

$$T_1^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad T_2^2=0,\quad T_3^2=0$$

$$T_1T_2+T_2T_1=0,\quad T_1T_3+T_3T_1=0$$

$$T_2T_3+T_3T_2=\mathbb{1}$$

$$\begin{aligned}
&X^0=\mathbb{1}\\
&X^1=X\\
&X^2=(x\,T_1+\cdots)^2=(x^2+y\,z)\mathbb{1}\equiv\theta^2\,\mathbb{1}\\
&X^3\sim X\\
&X^4\sim\mathbb{1}
\end{aligned}$$

2) Use Caley–Hamilton theorem: Every matrix $M$ satisfies its secular equation.

Example:

$$X=\begin{pmatrix} x & y\\ z & -x\end{pmatrix}$$

$$X-\lambda\mathbb{1}=\begin{pmatrix} x-\lambda & y\\ z & -x-\lambda\end{pmatrix}$$

Then $\det(X-\lambda\mathbb{1})=0$:

$$\lambda^2=x^2+y\,z\equiv\theta^2$$

$$\lambda^2=-\det X$$

So CH theorem: $X^2 = (x^2 + y\,z)\mathbb{1} = -\det X\,\mathbb{1}$.

In general for $2\times 2$ matrices $A$:

$$A^2 - A\,\mathrm{tr}\,A + \mathbb{1}\,\det A = 0.$$

Exercise: Find the expansion for $3\times 3$ matrices.

**Three important questions.**

$$\begin{array}{ccc} \text{Lie} & \xrightarrow{\text{linearization}} & \text{Lie} \\ \text{group} & \xleftarrow[\exp?]{} & \text{algebra} \end{array}$$

i) Does the exponential map give back the whole group?

ii) Are Lie groups with isomoprhic Lie algebras themselves isomorphic?

iii) Is the map Lie algebra $\xrightarrow{\exp}$ Lie group unique?

In general all answers are no!

i) For compact groups: Yes!

$$\begin{array}{lll} \mathrm{SU}(2)\text{: Yes} & x^2 + y^2 + z^2 + w^2 = 1 \ \text{ or } \ |\alpha|^2 + |\beta|^2 = 1 & \\ \mathrm{SL}(2,\mathbb{R})\text{: No} & \approx \mathrm{SU}(1,1)\text{:} \quad |\alpha|^2 - |\beta|^2 = 1 \quad \Rightarrow \quad x^2 + y^2 - z^2 - w^2 = 1 & \end{array}$$

But Cartan found that if we exponentiate using $\mathrm{e}^{X_{\text{compact}}}\,\mathrm{e}^{X_{\text{noncompact}}}$ it works. You can divide the generators into compact and non-compact ones and exponentiate them separately.

ii) No, since SU(2) and SO(3) have the same Lie algebra, but the groups are different. You can't exponentiate blindly. If one gets the universal covering group the answers is yes.

iii) Is not unique! But can be related by Campbell–Baker–Hausdorff formula (CBH).

**Lie algebras: general properties**

Vector space, Lie product, Jacobi identity.

*Comments:*

1) If $X$ and $Y \in \mathrm{Lie}(G)$ then so is $\alpha\,X + \beta\,Y$ (vector space) since

$$(\mathbb{1} + \varepsilon\alpha\,X + \cdots)(\mathbb{1} + \varepsilon\beta\,Y + \cdots) = \mathbb{1} + \varepsilon\underbrace{(\alpha X + \beta\,Y)}_{\in \mathrm{Lie}(G)} + \cdots$$

ii)

$$g_1 g_2 g_1^{-1} g_2^{-1} = (\mathbb{1} + \varepsilon_1 X + \cdots)(\mathbb{1} + \varepsilon_2 Y + \cdots) \times (\mathbb{1} - \varepsilon_1 X + \cdots)(\mathbb{1} - \varepsilon_2\,Y + \cdots) =$$

The linear terms cancel so the $X^2$ etc must be included.

$$= \mathbb{1} + \varepsilon_1\varepsilon_2 [X, Y] + \cdots$$

This means that $[X, Y]$ must also be in the Lie algebra. Then it must be expressible in terms of the generators too. With $X = x^a T^a$ we get

$$[T^a, T^b] = f^{a\,b}{}_c\,T^c$$

where the coefficients $f^{a\,b}{}_c$ are called *structure constants*.

iii) Jacobi identity. With $[x, y] = xy - yx$ it is trivial. But the product $X\,Y$ is never needed: when the product appears, it always appears as a commutator. So strictly speaking you never need the ordinary product $X\,Y$.

$$\mathrm{e}^X\,\mathrm{e}^Y \underset{\text{CBH}}{=} \mathrm{e}^{X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[Y,[Y,X]] + \frac{1}{12}[X,[X,Y]] + \cdots}$$

Example: Use it for Heisenberg:

$$[\hat{p}, \hat{q}] = -\mathrm{i}\,\hbar.$$