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Characters and the Great Orthogonality Theorem

We will use this to do polynomial equations and some quantum mechanics.

EXAMPLE: *Recall:* D_3, representation $\Gamma(R), \forall R \in G$. Matrices, g of them ($g = |G|$).

i) 1 (one-dimensional representation).

ii) 1 (E, D, F) and -1 (A,B,C) (one-dimensional representation)

iii) 2×2 matrix representation.

others: 3×3 matrices, obtained from viewing D_3 as the permutation group S_3.

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

where the numbers refer to the corners of the triangle.

also, later: the regular representation which is g-dimensional.

— *Goal:* Classify all irreducible representations of any group G. —

Characters: Traces of $\Gamma(R)$ in representation i: $\chi^{(i)}(R) = \mathrm{Tr}\big(\Gamma^{(i)}(R)\big)$. We had character tables, with "funny" orthogonality properties.

Recall: "Variant of Schur's lemma": If M exists such that for two irreducible representations $\Gamma^{(i)}$ and $\Gamma^{(j)}$ (dimensions l_i and l_j, respectively):

$$M\,\Gamma^{(i)}(A_k) = \Gamma^{(j)}(A_k)\,M \quad \text{for all } A_k \in G$$

then either (1) $l_i \neq l_j$ and $M = 0 \Leftrightarrow$ the representations are inequivalent; or (2) $l_i = l_j$ and $M = 0$ (the representations are inequivalent) or $\det M \neq 0$ (the representations are equivalent).

THEOREM: **The Great Theorem**

Consider all irreducible representations of a group G, $\Gamma^{(i)}$, then

$$\sum_{\forall R \in G} \big(\Gamma^{(i)}(R)\big)^*_{\mu\nu} \big(\Gamma^{(j)}(R)\big)_{\alpha\beta} = \frac{g}{l_i}\,\delta_{ij}\,\delta_{\mu\alpha}\,\delta_{\nu\beta}$$

Implications of this theorem

1.

$$\sum_i (l_i)^2 \leqslant g.$$

g is the number of components of the vectors that are orthogonal; the sum ranges over g terms, where g is the order of the group. $\sum_i (l_i)^2$ are the number of vectors enumerated by i, μ, ν.

We will later prove equality: $\sum_i (l_i)^2 = g$. That is the *dimensionality theorem*.

2. Turn the Great Theorem into a theorem for characters:

$$\chi^{(i)}(R) = \mathrm{Tr}\big(\Gamma^{(i)}(R)\big) = \sum_{\mu=1}^{l_i} \big(\Gamma^{(i)}(R)\big)_{\mu\mu}$$

$$\sum_{\substack{\mu \\ (\mu=\nu)}} \sum_{\substack{\alpha \\ (\alpha=\beta)}} \text{ on the Great orthogonality theorem implies}$$

$$\sum_R \big(\chi^{(i)}(R)\big)^* \chi^{(j)}(R) = \frac{g}{l_i} \sum_\mu \sum_\alpha \delta_{ij}\, \delta_{\mu\alpha}\, \delta_{\mu\alpha}$$

$$= \frac{g}{l_i}\, \delta_{ij} \sum_{\mu\alpha} \delta_{\mu\alpha}\, \delta_{\mu\alpha} = \frac{g}{l_i}\, \delta_{ij} \underbrace{\sum_\mu \delta_{\mu\mu}}_{=l_i} = g\, \delta_{ij}$$

Note: $\chi^{(i)}(R)$ are the same within each class ($x\, A\, x^{-1} = B$, trace cyclic). Thus

$$\sum_{\substack{k \\ \text{sum of} \\ \text{classes}}} \chi^{(i)}(\mathcal{C}_k)\, \chi^{(j)}(\mathcal{C}_k)\, N_k = g\, \delta_{ij}$$

where N_k is the number of elements in the class $\mathcal{C}_k$. Conclusion: orthogonality of *rows* in the character table, and the number of irreducible representations $\leqslant$ the number of classes. [Consider $\chi^{(i)}(\mathcal{C}_k)$ to be the k'th element of vector number i: then the vectors are orthogonal. The number of orthogonal vectors is always less than or equal to the dimension of the vector space.]

Also: Form the matrix

$$Q = \begin{pmatrix} \chi^{(1)}(\mathcal{C}_1) & \chi^{(1)}(\mathcal{C}_2) & \cdots \\ \chi^{(2)}(\mathcal{C}_1) & \ddots & \\ \vdots & & \end{pmatrix}$$

and consider

$$Q' = \frac{1}{g} \begin{pmatrix} \chi^{(1)}(\mathcal{C}_1)^*\, N_1 & \chi^{(2)}(\mathcal{C}_1)^*\, N_1 & \cdots \\ \chi^{(1)}(\mathcal{C}_2)^*\, N_2 & \ddots & \\ \vdots & & \end{pmatrix}$$

$$\Rightarrow \quad (Q\, Q')_{ij} = \sum_k \frac{\chi^{(i)}(\mathcal{C}_k)^* \chi^{(j)}(\mathcal{C}_k)\, N_k}{g} = \delta_{ij}$$

$$\Rightarrow Q' = Q^{-1}$$

$$\Rightarrow (Q' Q)_{ij} = \delta_{ij}$$

$$\Rightarrow \sum_i \chi^{(i)*}(\mathcal{C}_k) \chi^{(i)}(\mathcal{C}_l) = \frac{g}{N_k}\, \delta_{kl}$$

i.e. orthogonality of the columns of the character table.

So: the number of classes equals the number of irreducible representations.

EXAMPLE: D_3: $g = 6$, number of classes: 3. [This means that the number of irreducible representations is 3, which we can insert into the dimensionality theorem stated above.]

$$\Rightarrow \sum_{i=1}^{3} (l_i)^2 = 6.$$

Only one way of doing that: $l_1=1, l_2=1, l_3=2$.

Summary: Rules for construction of character tables.

i) number of irreducible representations = number of classes.

ii) $\sum_i (l_i)^2 = g$

iii) orthogonality of character columns.

iv) orthogonality of character rows.

v) another rule (not proved yet):

$$\big(N_j\chi^{(i)}(\mathcal{C}_j)\big)\big(N_k\chi^{(i)}(\mathcal{C}_k)\big) = l_i \sum_l c_{jkl}\, N_l\, \chi^{(i)}(\mathcal{C}_l)$$

where c_{jkl} are called *class multiplication constants*.

Exercise: Reconstruct the D_3 character table from properties i) to iv) above, and check v).

Furthermore: Decomposition of reducible representations.

Example: 3×3 representation of D_3

$$S\begin{pmatrix} & & \\ & 3\times 3 & \\ & & \end{pmatrix}S^{-1} \overset{?}{=} \left(\begin{array}{c|c} 1\times 1 & 0 \\ \hline 0 & 2\times 2 \end{array}\right)$$

For any representation $\Gamma(R)$ with characters $\chi(R)$ we have

$$\chi(R) = \sum_i a_i\, \chi^{(i)}(R) \quad \forall R.$$

But then

$$a_i = \frac{1}{g}\sum_k N_k \chi^{(i)*}(\mathcal{C}_k)\chi(\mathcal{C}_k)$$

(Exercise!)

Definition: Apply this to the *regular representation*:

Example: Write the multiplication table of D_3 as

D_3	E	A	B	C	D	F
E	E	A				
A^{-1}	A	E				
B^{-1}			E			A
C^{-1}				E	A	
D^{-1}				A	E	
F^{-1}			A			E

$\Rightarrow$ the g-dimensional representation is

$$\Gamma^{\text{reg}}(E) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \Gamma^{\text{reg}}(A) = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{pmatrix}$$

This is a g-dimensional representation.

Properties: $\chi^{\text{reg}}(E) = l^{\text{reg}} = g$, $\chi^{\text{reg}}(R)|_{R\neq E} = 0$. This implies a theorem:

THEOREM: **The celebrated theorem**

$a_i = l_i$, where the a_i are the expansion coefficients of the χ^{reg}.

Proof.

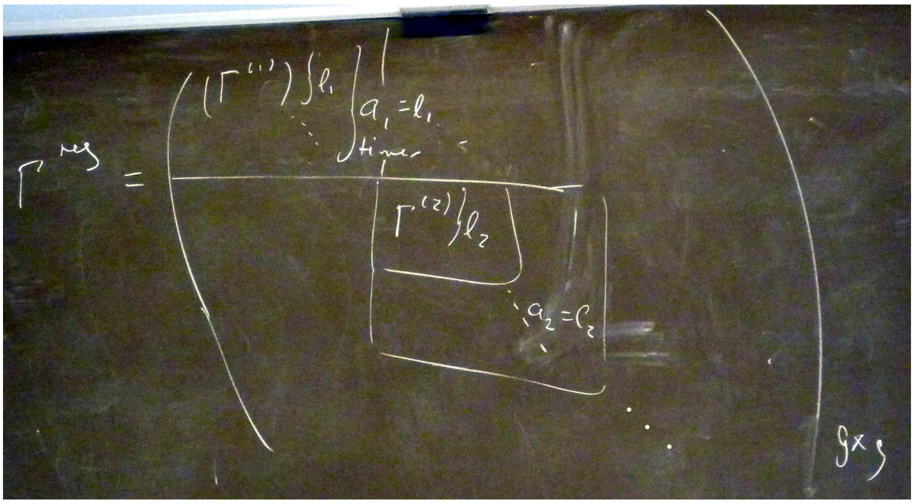
$$a_j = \frac{1}{g}\sum_R \chi^{j*}(R)\,\chi^{\text{reg}}(R) = \frac{1}{g}\chi^{(j)*}(E)\cdot\chi^{\text{reg}}(E) = l_j.$$

□

THEOREM: **The dimensionality theorem**

$$g = \sum_i (l_i)^2$$

Proof.



$$\Rightarrow g = \sum_i (l_i)^2$$

□

EXERCISE: For D_3, decompose the 3×3 representation of S_3 given previously.

EXERCISE: Find all irreducible representations of any finite group of prime order.

$$\text{prime order} \Rightarrow \text{cyclic} \Rightarrow \text{abelian} \Rightarrow \text{each element is its own class!} \Rightarrow \text{there are } g \text{ classes} \Rightarrow$$

$$\text{each representation is one-dimensional:} \sum_i (l_i)^2 = g.$$

$\exp\left(2\pi\,\mathrm{i}\,\frac{n}{g}\right)$, $n = 0, \ldots, g-1$.

Basic Galois theory

Consider an n:th order polynomial equation in $z \in \mathbb{C}$ with roots z_i, $i = 1, \ldots, n$.

$$(z - z_1)(z - z_2) \cdots (z - z_n) = 0.$$

$$z^n - I_1 z^{n-1} + I_2 z^{n-2} + \cdots + (-1)^n I_n = 0$$

where

$$\begin{cases} I_1 = z_1 + z_2 + \cdots + z_n \\ I_2 = \sum_{i<j} z_i z_j = z_1 z_2 + z_1 z_3 + \cdots + z_2 z_3 + \cdots \\ \vdots \\ I_n = z_1 z_2 \cdots z_n \end{cases}$$

These I's are all invariant under permutations of the z_i's, i.e. under the Galois group S_n.

Note: Any function $f(z_i)$ invariant under S_n can be expanded in terms of I's.

$$f^{\mathrm{inv}} = f(I_1, I_2, \ldots, I_n)$$

Problem: Find all z_i's expressed in terms of the I's. You know the I's from the polynomial equation — if you can express the z_i's in terms of the I's, you have solved the polynomial equation.

Galois theorem A solution $z_i \in \mathbb{C}$ can be found *iff* there is a chain of subgroups in the subgroup diagram of S_n such that each arrow in the chain conects a G and an H such that

i) H is an invariant subgroup of G,

ii) G/H is abelian.

This is *constructive* — i.e. this gives an explicit solution to the equation.

Solve the cubic

Galois group $= S_3$.

Diagram: figure

Is S_3/A_3 abelian? Yes, $S_3/A_3 \sim S_2$.

Also $A_3/I = A_3$ is abelian.

Polynomial equation $(z - r_1)(z - r_2)(r - r_3) = 0$

$$\begin{cases} I_1 = r_1 + r_2 + r_3 \\ I_2 = r_1 r_2 + r_1 r_3 + r_2 r_3 \\ I_3 = r_1 r_2 r_3 \end{cases}$$

These are all invariant under S_3. Now, start from the last double arrow.

$$A_3/I = A_3 \text{ is abelian, } \mathrm{order}(A_3) = 3.$$

This implies 3 one-dimensional irreducible representations.

A_3	I	(123)	(132)
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	ω	ω^2
$\Gamma^{(3)}$	1	ω^2	ω

, $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$

One-dimensional representation can be written from this

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 + r_3 \\ r_1 + \omega r_2 + \omega^2 r_3 \\ r_1 + \omega^2 r_2 + \omega r_3 \end{pmatrix}$$

That is

$$\Gamma^{(1)}: \quad v_1 = r_1 + r_2 + r_3$$

$\Rightarrow I, (123), (132)$ all give back v_1.

$$\Gamma^{(2)}: \quad v_2 = r_1 + \omega r_2 + \omega^2 r_3$$

Invariant under I and goes to ωv_2 under (123) and to $\omega^2 v_2$ under (132).

Note: Since v_1 is symmetric under S_3 and A_3 it must be expressible in terms of I's.

$$v_1 = I_1.$$

...while v_2 and v_3 are not invariant under A_3.

Next we turn to $S_3/A_3 = S_2$. We need functions invariant under A_3 but not under S_2. These are $(v_2)^3$ and $(v_3)^3$.

Check S_2. $(12) \in S_2$.

$$(12)(v_2)^3 = (r_2 + \omega r_1 + \omega^2 r_3)^3 = \underbrace{\omega^3}_{=1} (r_1 + \omega^2 r_2 + \omega r_3)^3 = (v_3)^3$$

Also: $(v_3)^3 \xrightarrow{(12)} (v_2)^3$.

This means that S_2 is the Galois group of a quadratic equation:

$$x \in \mathbb{C}: \quad (x - (v_2)^3)(x - (v_3)^3) = 0$$

Invariant under S_2:

$$\begin{cases} J_1 = (v_2)^3 + (v_3)^3 \\ J_2 = (v_2)^3 (v_3)^3 \end{cases}$$

These J_i's are invariant under $S_3 \Rightarrow$ They must be functions of the I's.

$$\begin{cases} J_1 = \displaystyle\sum_{i_1 + 2i_2 + 3i_3 = 3} A_{i_1 i_2 i_3} I_1^{i_1} I_2^{i_2} I_3^{i_3} \\ J_2 = \displaystyle\sum_{i_1 + 2i_2 + 3i_3 = 6} B_{i_1 i_2 i_3} I_1^{i_1} I_2^{i_2} I_3^{i_3} \end{cases}$$

But of course

$$\begin{Bmatrix} (v_2)^3 \\ (v_3)^3 \end{Bmatrix} = \frac{1}{2} J_1 \pm \sqrt{J_1^2 - J_2}$$

$$\Rightarrow \quad \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \left(\frac{1}{2} J_1 \pm \sqrt{J_1^2 - J_2} \right)^{1/3}$$

Finally, the solution is obtained by inverting the character table:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \text{character} \\ \text{table} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Home problem: quartic. figure.