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Are you doing the small exercises? You need to do them! It is very easy to misjudge your understanding here. Good boks: Gilmore "Lie Groups, Physics and Geometry", Tinkham — Wikipedia is good. The mathematics in Wikipedia is fantastic.

So far we have defined what we mean by a *group* G and we talked about *cosets* G/H, where H could be an *invariant* (or *normal*) subgroup. We talked about *classes*: $x\,A\,x^{-1} = B$. From this you form an invariant subgroup.

EXAMPLE: D_3:

$$E, \underbrace{A, B, C}_{\text{flips}}, \underbrace{D, F}_{\text{rotations}}$$

There are two types of subgroups here. We have $\{E, D, F\}$, and $\{E, A\}$, $\{E, B\}$, $\{E, C\}$. The first one, $\{E, D, F\}$, is invariant, but the others are not invariant. Let's call the rotatio subgroup $H_{\mathrm{R}} = \{E, D, F\}$.

$$G/H_{\mathrm{R}} = \text{group}$$

$$G/\{E, A\} = \text{"coset space", not a group}$$

Representations (matrices)

EXAMPLE: D_3:

1. $A_i \to 1$

   (holomorphic, many-to-one)

2. $\{E, D, F\} \to 1$

   $\{A, B, C\} \to -1$

   (holomorphic, many-to-one)

3. $A_i \to 2 \times 2$ matrices: faithful.

   (isomorphic, one-to-one)

4. $A_i \to 3 \times 3$ matrices: faithful.

S_n is the symmetry group of n elements = {all permutations}.

$$S_3 \approx D_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{etc}$$

which just permutate three elements, e.g.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \to \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

EXERCISE: find these matrices!

Note: There exists a subgroup A_3 of S_3 given by elements with det = +1.

Note: A_n, for $n \geqslant 5$, (order $|A_n| = \frac{1}{2}n!$).

Can we classify all irreducible representations?

Meaning: An irreducible representations ("irrep") cannot be brought to block-diagonal form by any change of basis.

Representation: $\Gamma(A)$ where A is the formal element in G, and Γ is a matrix of some *size*.

Irreducibility: a change of basis: $\Gamma \rightarrow S^{-1}\,\Gamma\,S$. This is a "similarity transformation".

EXAMPLE: Can representation 4 of D_3 be written in block form using smaller representations?

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \rightarrow S^{-1}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} S = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

then 4 is shown to be *fully reducible*.

Note: It could happen that the closest to block form possible is

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

This is called *reducible*, but not *fully reducible*.

EXERCISE: Write the Poincaré Lie algebra in this form.

To answer questiosn about the existence of irreducible representations we need a basis-independent concept: the character: $\Gamma^{(i)}(A) \rightarrow \chi^{(i)}\big(\Gamma^{(i)}(A)\big)$ where A is an element in G and i is the representation number. The definition is

$$\chi^{(i)}\big(\Gamma^{(i)}(A)\big) \equiv \mathrm{Tr}\big(\Gamma^{(i)}(A)\big)$$

which means that $\Gamma(A)$ and $S^{-1}\Gamma(A)\ S$ give the same $\chi^{(i)}(\Gamma(A))$. $\chi^{(i)}$ depends really only on *classes* $(x\,A\,x^{-1} = B)$.

Character table:

$$D_3:\quad \begin{array}{l} \mathcal{C}_1 = \{E\} \\ \mathcal{C}_2 = \{A, B, C\} \\ \mathcal{C}_3 = \{D, F\} \end{array} \quad \rightarrow \quad \begin{array}{l|lll} \chi^{(i)}(R) & \mathcal{C}_1 & 3\,\mathcal{C}_2 & 2\,\mathcal{C}_3 \\ \hline \Gamma^{(1)} & 1 & 1 & 1 \\ \Gamma^{(2)} & 1 & -1 & 1 \\ \Gamma^{(3)} & 2 & 0 & -1 \end{array}$$

Note two funny propertiets: The columns are orthogonal to each other. The rows are orthogonal using the number of elements (which we also write as coefficents in the label row) as a metric.

These facts follow from the Great Orhogonality Theorem.

Proof

First: prove that any matrix representation with det $A_i \neq 0$, $A_i \in G$ (denote matrix representation by A_i instead of $\Gamma(A_i)$ is equivalent (via a similarity transformation) to a unitary representation.

This follows from:

Consider *some* representation $A_i \in G$. Then form

$$H = \sum_{i=1}^{g} A_i A_i^{\dagger} = H^{\dagger} \quad \text{i.e. } H \text{ is hermitian.}$$

But any hermitian matrix can be diagonalised by a unitary matrix, i.e.

$$\exists U \text{ such that } d = U^{-1} H U, U U^{\dagger} = 1, d = \text{diagonal matrix.}$$

Exercise: show this!

Here all elements in d are real and positive. Positive, since

$$d = U^{-1} H U = \sum_{i=1}^{g} U^{-1} A_i A_i^{\dagger} U = \sum_{i=1}^{g} \underbrace{U^{-1} A_i U}_{\equiv A_i'} \underbrace{U^{-1} A_i^{\dagger} U}_{\equiv A_i'^{\dagger}} \equiv \sum_i A_i' A_i'^{\dagger}.$$

But then (no summation convention)

$$d_{\mu\mu} = \sum_i (A_i' A_i'^{\dagger})_{\mu\mu} = \sum_i \sum_{\nu} (A_i')_{\mu\nu} (A_i')_{\nu\mu} = \sum_i \sum_{\nu} (A_i')_{\mu\nu} (A_i'^{*})_{\mu\nu} = \sum_i \sum_{\nu} |A_i'|^2 > 0$$

$$\Rightarrow d^{1/2} \text{ is possible to form} \Rightarrow A_i'' = d^{-1/2} A_i' d^{1/2}$$

is unitary.

Check: $A_i'' A_i''^{\dagger} = d^{-1/2} A_i' d^{1/2} d^{1/2} A_i'^{\dagger} d^{-1/2}$. Insert $\mathbb{1} = d^{-1/2} \sum_i A_i' A_i'^{\dagger} d^{-1/2}$.

$$A_i'' A_i''^{\dagger} = d^{-1/2} A_i' d^{1/2} \left(d^{-1/2} \sum_j A_j' A_j'^{\dagger} d^{-1/2} \right) d^{1/2} A_i'^{\dagger} d^{-1/2} =$$

$$= d^{-1/2} \sum_j (A_i' A_j') \big(A_j'^{\dagger} A_i'^{\dagger} \big) d^{-1/2} = d^{-1/2} \sum_j (A_i' A_j') (A_i' A_j')^{\dagger} d^{-1/2} =$$

(rearrangement theorem)

$$= d^{-1/2} \sum_j A_j' A_j'^{\dagger} d^{-1/2} = \mathbb{1}.$$

$$= A_i'' \text{ are unitary.}$$

Schur's lemma

All matrices that commute with all the matrices in an irreducible representation must be constant, as in proportional to the unit matrix.

Proof. From the previous statement we can always use a unitary representation. So let $A_i \in G$ (really $\Gamma(A_i)$) be unitary. Then the matrix M satisfies $M A_i = A_i M, \forall A_i \in G$. Then $\dagger$ implies:

$$A_i^{\dagger} M^{\dagger} = M^{\dagger} A_i^{\dagger}$$

$A_i (...) A_i \Rightarrow$

$$M^{\dagger} A_i = A_i M^{\dagger}.$$

Thus, if M commutes with all A_i's, so does $M^{\dagger}$ and so do $H_1 = M + M^{\dagger}$ and $H_2 = \mathrm{i}\,(M - M^{\dagger})$ which are both hermitian.

So, next we prove that a with all A_i commuting hermeitian matrix is *constant*. Now, since H is hermitian (both of them), it is diagonalisable:

$$U \text{ exists such that } d = U^{-1} H U$$

$$\Rightarrow \quad A_i' d = d A_i' \quad \text{for } A_i' = U^{-1} A_i U.$$

Check: $U^{-1} A_i U U^{-1} H U = U^{-1} H U U^{-1} A_i$. OK.

In components $A_i d = d A_i$ reads

$$(A_i')_{\mu\nu} d_{\nu\nu} = d_{\mu\mu} (A_i)_{\mu\nu}$$

$$\Rightarrow (A_i')_{\mu\nu} (d_{\nu\nu} - d_{\mu\mu}) = 0.$$

So if $d_{\nu\nu} \neq d_{\mu\mu}$ then $(A_i')_{\mu\nu} = 0$ for all i.

Example. Supose

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \forall i: \quad A_i = \begin{pmatrix} \neq 0 & 0 \\ 0 & \neq 0 \end{pmatrix}$$

but then the rperesentation is *reducible*.

Also if A_i is an irreducible representation then all $d_{\mu\mu}$ must be equal. $d = k\,\mathbb{1}$.

Lemma (variant of Schur)

If M exists such that for two irreducible representations $\Gamma^{(i)}$ and $\Gamma^{(j)}$ of dimension l_i and l_j and M is such that it commutes:

$$M\,\Gamma^{(i)}(A_k) = \Gamma^{(j)}(A_k)\,M, \quad \forall k$$

Then: (1) $l_i \neq l_j$ and $M = 0 \Rightarrow$ The representations are inequivalent. Or

(2) $l_i = l_j$ and $M = 0$ (irreducible representations inequivalent) or $\det M \neq 0$ (irreducible representations equivalent).

Finally,

The Great Orthogonality Theorem

Consider *all inequivalent irreducible representations, unitary* $\Gamma^{(i)}(R)$ of a group G, then

$$\sum_{\forall R \in G} (\Gamma^{(i)}(R))^*_{\mu\nu} \big(\Gamma^{(j)}(R)\big)_{\alpha\beta} = \frac{g}{l_i}\,\delta_{ij}\,\delta_{\mu\alpha}\,\delta_{\nu\beta}.$$

The orthogonality here is between $(\text{vectors})_{i\mu\nu} = (\Gamma^{(i)}(A_1)_{\mu\nu}, \Gamma^{(i)}(A_2)_{\mu\nu}, \ldots, \Gamma^{(i)}(A_g)_{\mu\nu})$. How many vectors? (number of irreps) $\times\, l_i^2$.

Proof. First, consider two inequivalent irreducible representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ and form

$$M = \sum_{R \in G} \Gamma^{(2)}(R)\,X\,\Gamma^{(1)}(R^{-1})$$

Here X is any matrix. Then M satisfies

$$M\,\Gamma^{(1)}(R) = \Gamma^{(2)}(R) M$$

(for any X).

Exercise: Check this! (Rearrangement theorem needed.)

But then $M = 0$ by previous lemma.

$$\Rightarrow M_{\alpha\mu} = 0 = \sum_R \sum_{\bar{\beta},\bar{\nu}} (\Gamma^{(2)}(R))_{\alpha\bar{\beta}} X_{\bar{\beta}\bar{\nu}} \big(\Gamma^{(1)}(R^{-1})\big)_{\bar{\nu}\mu}.$$

Now, since X is arbitrary set $X_{\bar{\beta}\bar{\nu}} = 1$ (let this be the only non-zero element). Then we get (skipping the bar on $\bar{\beta}$ and $\bar{\nu}$):

$$\sum_R (\Gamma^{(2)}(R))_{\alpha\beta} (\Gamma^{(1)}(R))^*_{\nu\mu} = 0$$

This is the δ_{ij} on the right hand side with $i \neq j$, since we assumed that the representations are different.

Secondly: Let $i = j$. Now we look at the same representation.

$$M = \sum_R \Gamma(R) X\, \Gamma(R^{-1})$$

By Schur's lemma

$$M = c\,\mathbb{1}.$$

$X_{\nu\rho} = 1 \Rightarrow$

$$c_{\nu\rho} \delta_{\mu\mu'} = \sum_R (\Gamma(R))_{\mu\nu}\, (\Gamma(R^{-1}))_{\rho\mu'}.$$

$$c_{\nu\rho}\, l = \sum_R \sum_\mu (\Gamma(R))_{\mu\nu}\, (\Gamma(R^{-1}))_{\rho\mu} = \sum_R (\Gamma(R^{-1})\Gamma(R))_{\rho\nu} =$$

$$= \sum_R (\Gamma(R^{-1}R))_{\rho\nu} = \sum_R (\Gamma(\mathbb{1}))_{\rho\nu} = \sum_R \delta_{\rho\nu} = g\,\delta_{\rho\nu}$$

$$c_{\nu\rho} = \frac{g}{l}\,\delta_{\nu\rho}$$

Thus

$$\sum_R (\Gamma^{(i)}(R))^*_{\mu\nu}\, \big(\Gamma^{(j)}(R)\big)_{\alpha\beta} = \frac{g}{l_i}\,\delta_{ij}\,\delta_{\mu\alpha}\,\delta_{\nu\beta}$$