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Groups

Definition: Group G, elements $g \in G$ (for finite groups we also use capital letters $A, B, C, \ldots$ for the group elements). There is a composition operator ("multiplication"), denoted $\cdot$ or $\circ$. A group satisfies the axioms

1. Closure: $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 = g_3 \in G$.
2. Associativity: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$. This is satisfied by matrices, but there are objects that do not satisfy associativity, e.g. octonions.
3. Unit element: $\exists E \colon E\, A_i = A_i \forall A_i \in G$.
4. Inverse: For each $A_i \exists A_i^{-1}$ such that $A_i A_i^{-1} = E$.

There are many kinds of groups:

- Finite groups (discrete groups).
- Lie groups: infinitely many elements, continuous. Either finite dimensional groups or infinite dimensional groups. The dimension is the number of continuous parameters.

There is also something called supergroups. There are also quantum groups, but they are not groups at all.

We had one example last time, which we called D_3, which is the symmetry group of an equilateral triangle.

Figure 1. A, B, C flip elements, D, F are rotations, E unit operaton. This leads to a multiplication table.

Note: There exists a matrix representation of the multiplication table by 2×2 matrices: *faithful*, or *true* (one-to-one: isomorphism).

You can put all elements equal to one: that trivially satisfies the multiplication table. You can also do $E, D, F = 1$ and $A, B, C = -1$. These are many-to-one: homomorphism.

Big question: Are there any other matrix representations? (What exactly is the question?) There is an answer for a well-formulated question here.

Definition: A *representation* (here we always refer to matrix representations) satisfies

$$A, B, C \in G \quad \Rightarrow \quad A \xrightarrow{\Gamma} \Gamma(A) \text{ etc such that if } A\, B = C \text{ then } \Gamma(A)\, \Gamma(B) = \Gamma(C).$$

We proved the *rearrangement theorem* last time.

Cosets: A *subgroup* H in G is a subset satisfying the group axioms.

Consider $H x$. If $x \in H \Rightarrow H x = H$. If $x \notin H \Rightarrow H x =$ set disjoint to H.

Figure 2. Set of disjoint subsets called cosets. It is a coset space (the "space" part makes more sense when we come to Lie groups).

$H x$ is called a right coset. $H \backslash G$. Similarly for $x\, H \rightarrow G / H$ left coset.

Two cosets are either identical or disjoint. $\Rightarrow |G| = g$ is the order of G, $|H| = h$, then $g = h \cdot l$ where l is an integer called the index of H in G. l is the number of cosets.

Classes: Two elements A and B are *conjugate* to each other if $A = x\,B\,x^{-1}$ for *some* $x \in G$.

Then if A is conjugate to B which is conjugate to C, then A is conjugate to C.

$$D_3\colon \qquad \underbrace{E, D, F,}_{H = \text{subgroup}} \; A, B, C$$

Subgroups: i) $H_{\text{i}}\{E, D, F\}$ of order 3; ii) $\{E, A\}, \{E, B\}, \{E, C\}$ have order 2.

$G/H = \text{group}\,(?)$

$G/H_{\text{i}} = \text{group}$. $G/H_{\text{ii}} \neq \text{group}$. Just a "coset set".

Classes in D_3

$$\mathcal{C}_1 = \{E\}, \quad \underbrace{\mathcal{C}_2 = \{A, B, C\}}_{\text{flips}}, \quad \underbrace{\mathcal{C}_3 = \{D, F\}}_{\text{rotations}}$$

Invariant subgroups

Let H be a subgroup of G and satisfy $x\,H\,x^{-1} = H$ for any $x \in G$.

First $x\,H = H\,x$.

DEFINITION: G is *simple* if and only if invariant subgroups are $\{E\}$ and G itself.

Note:

1. $H \cdot H = H$, where $H \cdot H \equiv \{h_1 \cdot h_2 |\, h_1 \in H, h_2 \in H\}$.

2. Let x_a be an element in G not in H (i.e. in $G/H = H\backslash G$) i.e. $H\,x_a = x_a H$. Then let $H\,x_a \equiv T_a$ so that $H\,T_a = H\,H\,x_a = H\,x_a = T_a$. Acts like a unit element.

3. $T_a \cdot T_b = H\,x_a\,H\,x_b = H\,H\,x_a\,x_b = H\,\underbrace{x_a\,x_b}_{x_{a\,b}} \equiv T_{a\,b}$. $\Rightarrow$ If H is invariant, then $G/H = H\backslash G$ is a group, called the factor group or the quotient group.

Class multiplication

If you take a class $\mathcal{C}_i$ with another class $\mathcal{C}_j$ keeping track of the elements and how many times they appear, the product $\mathcal{C}_i\mathcal{C}_j$ becomes a new class.

$$\mathcal{C}_i\mathcal{C}_j = \sum_k c_{i\,j\,k}\,\mathcal{C}_k.$$

Now we count the number of elements on each side, and they must match.

In D_3: $\mathcal{C}_2 = \{A, B, C\}$.

$$\mathcal{C}_2\,\mathcal{C}_2 = 3\,\mathcal{C}_1 + 3\,\mathcal{C}_3$$

$$\mathcal{C}_2\mathcal{C}_3 = 2\,\mathcal{C}_2$$

Let us enumerate some **examples of finite groups** to set the stage.

1. Order $= 1$: $\{E\}$.

2. Order $= 2$: $\{E, A\}$ such that $A^2 = E$.

3. Order = 3: $\{E, A, B\}$ where $B = A^2$.

4. Order = 4: only two cases. The continuation of the previous pattern: cyclic groups.

$$\{E, A, A^2, A^3\} \quad \text{where } A^4 = E.$$

The other case is the *Vierergruppe*. D_2: dihedral group.

$$\begin{array}{ccccc} & E & A & B & C \\ E & E & A & B & C \\ A & A & E & C & B \\ B & B & C & E & A \\ C & C & B & A & E \end{array} \quad \rightarrow \quad \text{abelian: } A\,B = B\,A \,\text{etc.}$$

Interpretation: Cyclic: rotations of □. Vierergruppe: cube with only π rotations. (Including $\pi/2$ rotations leads to a non-abelian group.)

Note $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\times$ is the direct product, i.e. $G = G' \times G''$, then with $(g_1', g_1'') \in G$ and (g_2', g_2'') then $(g_1', g_1'') \cdot (g_2', g_2'') = (g_1' \cdot g_2', g_1'' g_2'') \in G$.

Exercise: Verify from the table that $D_2 = \mathbb{Z}_2 \cdot \mathbb{Z}_2$.

Order = 5 (as for all prime orders):

- The group is unique and simple, and cyclic. $\{E, A, A^2, \ldots, A^4\}$ with $A^5 = E$.

This generalizes to all prime numbers.

Order = 6. Here we have the permutation group of three objects: symmetry group S_3. A particular subgroup is $A_3 =$ alternating subgroup $= \{$all even permutations$\}$.

Representation by 3×3 matrices:

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \rightarrow S^3. \quad \begin{array}{c} \text{if} \det(A) = +1 \\ \text{then } A \in A_3. \end{array}$$

Subgroup diagrams.

Figure 3.

Galois: If there is a path where all subgroups are invariant and G/H is abelian at each step starting from S_n any nth order oplynomial can be uniquely solved. The subgroup diagram for S_4 implies that any 4th order polynomial can be solved.

Finite groups in physics

Three dimensional lattices: covering operations.

- Translations: $T = n_1\, \boldsymbol{a}_1 + n_2\, \boldsymbol{a}_2 + n_3\, \boldsymbol{a}_3$ where $\boldsymbol{a}_i$ are the primitive vectors defining the lattice, and $n_i \in \mathbb{Z}$. This is an infinite order group.

- Rotations and inversions. Point groups: 32 examples.

Together these are called space groups, and there are 230 examples.

- In condensed matter

- Quantum Mechanics
- String theory (orbifolds)

There is a complete classification of all simple groups:

4 infinite series:

1. cyclic groups of prime order,
2. alternating groups of order $\geqslant 5$.
3. Lie groups of finite fields.
4. Tits groups.

There are also 26 sporadic cases. The most complicated one is the *Monster*.

Order $|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot \cdots \cdot 71 \approx 10^{55}$. Found by Griess in 1982. This group can be understood from the Leech lattice: in 24 dimensions: has 196883 nearest neighbours in the solid-state meaning. The automorphism group of this is related to the Monster. (This can be done in Conformal Field Theory and String theory.)

Chapter 2: Representations and characters.

Similarity transformations:

$$\Gamma_i(A) \to \Gamma_i'(A) = S^{-1}\,\Gamma_i\,S$$

$$\text{then} \quad \Gamma_i\,\Gamma_j = \Gamma_k \quad \Rightarrow \quad \Gamma_i'\,\Gamma_j' = \Gamma_k'$$

Then the representations Γ and Γ' are called *equivalent*. Also: If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are two representations, then of course

$$\begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix}$$

is also a representation. But this representation is *fully reducible*.

Note:

$$S^{-1}\begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix} S$$

can look very complicated in a general basis. That makes it hard to see that the representation is reducible.

If a general representation Γ can at best be brought to the form

$$\begin{pmatrix} \Gamma' & \Gamma'' \\ 0 & \Gamma''' \end{pmatrix}$$

then it is called *reducible*. If no simplifying basis exists then it is *irreducible*. The irreducible representations are the building blocks that we are normally interested in.

Question: How can we tell if a given representation Γ is irreducible or not?

Answer: Start by classifying all possible irreducible representations of the group.

Suppose

$$\Gamma = S^{-1}\begin{pmatrix} \Gamma' & 0 \\ 0 & \Gamma'' \end{pmatrix} S$$

The *character* is independent of the basis (independent of S). The charaacter is a set of g numbers ($i =$ representation) $\chi^{(i)} = \left\{ \chi^{(i)}(E), \chi^{(i)}(A_2), \ldots, \chi^{(i)}(A_g) \right\}$.

$$R \in G: \quad \chi^{(i)}(R) = \mathrm{Tr}\big(\Gamma^{(i)}(R)\big)$$

Comment: D_3: we know 4 different representations already $\{1\}$, $\{1, -1\}$, $\{2 \times 2 \text{ matrices}\}$, $\{3 \times 3\,\text{matrices}\}$.