2011–12–16

Lecturer: Bengt E W Nilsson

We do tuesdays 10 o'clock. Next week too! One lecture a week. 7.5 credits. Runs through all of the spring semester.

You need to do some homework to pass the course. Roughly one every two weeks. There will be small exercises in the lectures, that you're supposed to do. Do discuss within the group, otherwise it will be very hard.

Physics will always be in the back of our minds during the course, but we will not solve regular physics problems in the course.

Chapter 1. Introduction.

DEFINITION: A *group* G is a set of elements $g \in G$. (The set G can be of a number of varieties: it can be finite in number, it can be countably infinite, it can be continuous. The elements g themselves can be formal, they can be matrices, they can be differential operators, they can be actions on physical objects.)

These elements fullfil the axioms:

1. If $g_1 \in G$ and $g_2 \in G$, then $g_3 = g_2 \cdot g_1 \in G$. The dot is called *composition*. We will call it multiplication most of the time.
2. The multiplication is *assiciative*, i.e. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$. (Matrices, of course, satisfy this — and there are objects that don't satisfy it, such as octonions.)
3. There exists a *unique* element $e \in G$, called the *unit element*, such that $e \cdot g = g \cdot e = g$ for all $g \in G$.
4. For any $g \in G$ there exists a *unique* inverse g^{-1} such that $g^{-1} \cdot g = e = g \cdot g^{-1}$.

EXERCISE 1: Show that in axiom 4 above, one can demand $g \cdot g^{-1} = e$ and derive $g^{-1} \cdot g = e$.

EXERCISE 2: Show that the uniqueness of g^{-1} can be derived instead of postulated. (So the axiom is really about the existence of g^{-1}, as the uniqueness can be proven.)

EXERCISE 3: Rewrite $(g_1 \cdot g_2)^{-1}$ using g_1^{-1} and g_2^{-1}.

Overview of groups

Without defining them (for now) we will talk about *simple groups*. They cannot be divided into smaller groups (in some sense that we will discuss later on). They can be completely classified (in most cases). You can construct, from these simple groups, (almost) all groups.

A. Finite groups (or discrete groups)

The number of elements in a group G is called the order of the group and is denoted $|G|$. In this context we will often use capital letters for the group elements:

$$G = \{A_1, A_2, \ldots, A_{|G|}\}$$

$\rightarrow$ classification of $|G| < \infty$ groups:

There are 4 infinite series.

There are 26 sporadic groups. The most complicated, of all finite groups, is called the *Monster*.

EXAMPLE: $\mathrm{SL}(2, \mathbb{Z})$. The 2 means that we have 2×2 matrices. The L means *linear*, i.e. matrices. The S says that the determinant is one, $\det g = 1$, and $\mathbb{Z}$ says that the elements in the matrices belong to the integers $\mathbb{Z}$.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{cases} a, b, c, d \in \mathbb{Z}, \\ \det g = 1. \end{cases}$$

This is an infinite group.

Any use of it?

Moduli spaces (important in string theory in particular). Lead to modular functions, which can be used to describe non-perturbative effects in Quantum Field Theory (instantons, monopoles, black holes, et cetera).

Lattices (groups of this kind).

EXAMPLE: The Monster in three-dimensional quantum gravity has to do with black holes.

B. Lie groups

They have a continuous set of elements, parametrized by a number of continouous parameters. The number of continuous parameters is called the dimension $d = \dim(G)$.

B1. d can be finite. Then we have the Cartan classification:

Classical groups: A_n, B_n, C_n, D_n for $n \in \mathbb{Z}^+$ (positive integers).

Exceptional cases: E_6, E_7, E_8, G_2, F_4.

The indices here are called the rank and are related to d, the dimension.

B2. d infinite.

Affine (in two-dimensional solid state systems this leads to a classification of critical exponents). 3d: Thurston's classification of 3-manifolds.

Virasoro.

Kac–Moody (KM).

KM $\rightarrow$ affine, hyperbolic, Lorentzian $\rightarrow$ Borcherds.

C. Supergroups

For example, supersymmetry.