2009–12–09

Lecturer: Bengt E W Nilsson

Fuchs and Schweigert, chapter 7:

Simple and affine Lie algebras

We want to classify simple Lie algebras.

Recall the structure of a general Lie algebra (finite-dimensional).

$$\underbrace{\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots}_{\text{semi-simple}} \oplus \mathfrak{g}^{\text{abelian}} \oplus \mathfrak{g}^{\text{nilpotent}}$$

$\mathfrak{g}_1$ is simple.

Figure 1.

The classification of simple Lie algebras is done here using the Cartan matrix. This will then give us all possible root systems. We use the Chevalley basis:

$$\Phi_{\text{s}}(\mathfrak{g}) = \left\{ \alpha^{(i)} \colon i = 1, \ldots, r = \text{rank} \right\}$$

$$\Phi(\mathfrak{g}) = \text{all roots}$$

Theorem: Φ_{s} uniquely gives the Lie algebra $\mathfrak{g}(\Phi_{\text{s}})$.

Proof: By actual construction.

Starting point: The Chevalley–Serre relations: ($r = \text{rank}$)

• $3\,r$ generators $\{H^i, E^i_{\pm}\colon\ i = 1, \ldots, r\}$. Sometimes you call them $\{h^i, e^i, f^i\}$. The $3r$ generators satisfy:

i)

$$\left[H^i, H^j\right] = 0\colon \quad \text{Cartan sub-algebra (CSA)}$$

ii)

$$\left[H^i, E^j_{\pm}\right] = \pm\, A^{ji} E^j_{\pm}, \quad \left[E^i_+, E^j_-\right] = \delta^{ij} H^i$$

This (ii) defines a number of $\mathfrak{sl}(2, \mathbb{R})$ sub-algebras, one for each i.

iii) The Serre relations (multi-commutators).

$$\left(\text{ad}_{E^i_{\pm}}\right)^{1 - A^{ji}} E^j_{\pm} = 0$$

That is,

$$\underbrace{\left[E^i_{\pm}, \left[E^i_{\pm}, \ldots, \left[E^i_{\pm}\left[E^i_{\pm}, E^j_{\pm}\right]\right] \ldots\right]\right]}_{\text{number of commutators: } 1 - A^{ji}} = 0$$

The root string from $E^j_{\pm}$ along the direction $E^i_{\pm}$ has $1 - A^{ji}$ nodes. This is called a *presentation modulo relation*. So A^{ij} is the only information used.

7.2. The Cartan matrix

Recall

$$A^{ij} = \frac{2\left(\alpha^{(i)}, \alpha^{(j)}\right)}{\left(a^{(j)}, a^{(j)}\right)} \equiv \left(\alpha^{(i)}, \breve{\alpha}^{(j)}\right)$$

Note: we change to the notation of Fuchs and Schweigert: $\alpha^{(i)} \cdot_{\varkappa} \alpha^{(j)} \equiv (\alpha^{(i)}, \alpha^{(j)})$.

Comment: If A^{ij} can be made block diagonal by renaming the $\alpha^{(i)}$'s then the algebra is not simple.

Properties of the Cartan matrix:

- $\forall i$: $A^{ii} = 2$, no sum over i. (You can always arrange it in this way.)
- If $A^{ij} = 0$ $(i \neq j)$ then also $A^{ji} = 0$, from the definition.
- $A^{ij} \in \mathbb{Z}^{(\leqslant 0)}$ for $i \neq j$.

Proof of the last point:

Consider a root string for any two roots (i.e. not only simple ones): I enumerates all roots

$$\left\{\alpha^{(I)} + m\alpha^{(J)}:\ m = -n_-, -n_- + 1, \ldots, n_+ - 1, n_+:\ n_- \in \mathbb{N}^+, n_+ \in \mathbb{N}^+\right\}$$

Here n_- and n_+ are finite integers since this root string is in some representation of an $\mathfrak{sl}(2, \mathbb{R})$ sub-algebra.

$$\Rightarrow \text{dimension of the representation} = n_+ + n_- + 1 = \Lambda + 1$$

Then since $\Lambda_{\text{lowest weight}} = -\Lambda_{\text{heighest weight}}$.

$$(N, N-2, \ldots, -N \rightarrow \dim = N + 1)$$

Taking the scalar product with $\breve{\alpha}^{(j)}$.

$$\Rightarrow \left(\alpha^{(I)} - n_-\alpha^{(J)}, \breve{\alpha}^{(J)}\right) = -\left(\alpha^{(I)} + n_+\alpha^{(J)}, \breve{\alpha}^{(J)}\right)$$

$$\Rightarrow 2\left(\alpha^{(I)}, \breve{\alpha}^{(J)}\right) = (n_- - n_+)\left(\alpha^{(J)}, \breve{\alpha}^{(J)}\right)$$

Now consider simple roots only: $\left(\alpha^{(j)}, \breve{\alpha}^{(j)}\right) = A^{jj} = 2$.

$$\left(\alpha^{(i)}, \breve{\alpha}^{(j)}\right) = n_- - n_+ \in \mathbb{Z}$$

Also

$$\text{if} \quad \left(\alpha^{(i)}, \breve{\alpha}^{(j)}\right) > 0 \quad \Rightarrow \quad n_- > n_+ \geqslant 0$$

$$\text{and if} \quad \left(\alpha^{(i)}, \breve{\alpha}^{(j)}\right) < 0 \quad \Rightarrow \quad n_+ > n_- \geqslant 0$$

Now we know that $\alpha^{(i)} - \alpha^{(j)}$ is never a root. $\Rightarrow n_- = 0 \Rightarrow$ first case is not possible. Thus $\left(\alpha^{(i)}, \breve{\alpha}^{(j)} \leqslant 0\right) \Rightarrow A^{ij} \leqslant 0$ for $i \neq j$.

Let us now use this result a little bit more generally:

Let A^{ij} be a generalised Cartan matrix; generalised in the sense that there is no condition on $\det A$.

$$
\begin{aligned}
A_2\colon\ & \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3\\
B_2\colon\ & \det\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} = 2\\
G_2\colon\ & \det\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = 1
\end{aligned}
$$

- $A^{ii} = 2$.
- $A^{ij} = 0 \Leftrightarrow A^{ji} = 0,\ i \neq j$.
- $A^{ij} \in \mathbb{Z}_{(\leqslant 0)}, i \neq j$
- $\det A$:

  a) $> 0 \Rightarrow$ finite-dimensional,

  b) $= 0 \Rightarrow$ affine: infinite-dimensional algebra.

  c) $< 0 \Rightarrow$ hyperbolic (Lorentzian futher down here).

A^{ij} is symmetrisable: We might put $A^{ij} \to (\alpha^{(i)}, \alpha^{(j)}) = \varkappa^{ij}$ symmetric.

With $\det \varkappa > 0 \Rightarrow$ Euclidean signature.

In particular, when $\det \varkappa = 0$ we get degenerate signature.

When $\det \varkappa < 0 \Rightarrow$ Lorentzian signature.

Consider now case (a), the finite-dimensional ones. Consider two vectors, α, β. The scalar product: $\alpha \times \beta = \cos\theta \sqrt{(\alpha, \alpha)\,(\beta, \beta)}$.

$$\left(\alpha, \check{\beta}\right) = 2\cos\theta \sqrt{\frac{(\alpha, \alpha)}{(\beta, \beta)}}$$

$$\left(\alpha, \check{\beta}\right)(\check{\alpha}, \beta) = 4\cos^2\theta \leqslant 4$$

i.e. $\alpha = \alpha^{(i)}, \beta = \alpha^{(j)}$.

$$A^{ij} A^{ji} \in \{0, 1, 2, 3, 4\}$$

For the value 4, $\cos^2\theta = 1 \Rightarrow \cos\theta = \pm 1$. $\theta \in \{0, \pi\} \Rightarrow \alpha^{(i)} = \pm\, \alpha^{(j)}$. The roots are the same, they don't span the root diagram.

For the other values:

1)

$$0 \Rightarrow A^{ij} = A^{ji} = 0 \quad \Rightarrow \quad \left(\begin{array}{c|c} \begin{matrix} 2 & \\ & 2 \end{matrix} & \\ \hline & \end{array}\right)$$

2) $A^{ij} = A^{ji} = -1 \Rightarrow \theta = 2\pi/3$.

3) $A^{ij} = -1, A^{ji} = -2 \Rightarrow \theta = 3\pi/4$.

4) $A^{ij} = -1,\ A^{ji} = -3 \Rightarrow \theta = 5\pi/6$.

We have all rank 2. How do we find higher rank ones?

Rank 3.

$$A^{ij} = \left(\begin{array}{c|cc} 2 & \alpha & \beta \\ \hline \alpha & & \\ \beta & & \tilde{A}_2 \end{array} \right)$$

where $\tilde{A}_2$ is rank 2 (one of them).

Useful formula:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & BD^{-1} \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(A - BD^{-1}C\right) \det D$$

So a finite dim rank 3 must have $\det A_3^{ij} > 0$.

Consider now an infinite-dimensional algebra:

$$A^{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad \det A = 0.$$

This is the affine Lie algebra called $A_1^{(1)}$. The subscript is the starting finite-dimensional Lie algebra. The superscript in brackets is the type of affinisation: (1), (2) or (3).

CSA $= \{H^0, H^1\}$. H^0 is the new affine CSA element.

$$[H^0, H^1] = 0$$

$$\left[E_+^0, E_-^0\right] = H^0, \quad \left[E_+^1, E_-^1\right] = H^1$$

$$\left[H^0, E_\pm^0\right] = \pm 2E_\pm^0, \quad \left[H^1, E_\pm^1\right] = \pm 2E_\pm^1$$

etc from A^{ij}.

Serre relations:

$$0 = \left[E_\pm^1, \left[E_\pm^1, \left[E_\pm^1, E_\pm^0\right]\right]\right]$$

and the same for 1 and 2 interchanged. We will draw the roots as follows.

Figure 2. $\alpha^{(1)} \equiv \alpha, \alpha^{(0)} = \delta - \alpha$

Serre:

$$\begin{cases} \alpha^{(0)} \\ \alpha^{(0)} + \alpha^{(1)} = \delta \\ \alpha^{(0)} + 2\alpha^{(1)} = \delta + \alpha \end{cases}$$

$\alpha^0 + 3\alpha^{(1)} = \delta + 2\alpha$ not a root, by Serre

Figure 3. Never stops, up- or downwards.

$$\Rightarrow \Phi(A_1^{(1)}) = \{\pm\alpha + n\,\delta\colon\ n \in \mathbb{Z}\} \cup \{n\delta\colon\ n \in \mathbb{Z} \setminus \{0\}\}$$

(Only roots. The origin is just $\{H^0, H^1\}$.)

How for a general Cartan matrix can we see if it can be made block diagonaly by renaming the roots?

Example:

$$\left(\begin{array}{c|cc} 2 & -2 & -1 \\ \hline -1 & 2 & 0 \\ -1 & 0 & 2 \end{array}\right)$$

Is this the same as the following?

$$\left(\begin{array}{c|cc} 2 & -1 & -2 \\ \hline -1 & 2 & 0 \\ -1 & 0 & 2 \end{array}\right)$$

Yes. $(2 \leftrightarrow 3)$

Can we design a method that makes this obvious? That is the Dynkin diagram.

Rules:

— Each simple root $\alpha^{(i)}$ gives a node. $\circ$

— i) $A^{ij} = A^{ji} = -1$ connects two nodes with one line. $i \circ\!\!-\!\!\!-\!\!\circ\, j$

ii) $A^{ij} = -1, A^{ji} = -2$ gives $i \circ \!=\!\Leftarrow\! \circ\, j$. There is a direction in this case:

$$A^{ij} = \frac{2\left(\alpha^{(i)}, \alpha^{(j)}\right)}{\left(\alpha^{(j)}, \alpha^{(j)}\right)} = -1$$

$$A^{ji} = \frac{2\left(\alpha^{(j)}, \alpha^{(i)}\right)}{\left(\alpha^{(i)}, \alpha^{(i)}\right)} = -2$$

$$\Rightarrow 2 = \frac{\left(\alpha^{(j)}, \alpha^{(j)}\right)}{\left(\alpha^{(i)}, \alpha^{(i)}\right)} \Rightarrow |\alpha^{(j)}| > |\alpha^{(i)}|$$

Figure 4.

7.4. Simple finite Lie algebras

A_r $\mathfrak{sl}(r+1)$

B_r $\mathfrak{so}(2\,r+1)$

C_r $\mathfrak{sp}(r)$

D_r $\mathfrak{so}(2\,r)$

Figure 5.

These are called the classical ones. These are all realized by matrices: Then we have the exceptional ones:

E_6, E_7, E_8. Then it stops. F_4, G_2.

Figure 6.

E_8 manifold on the net.

Figure 7.