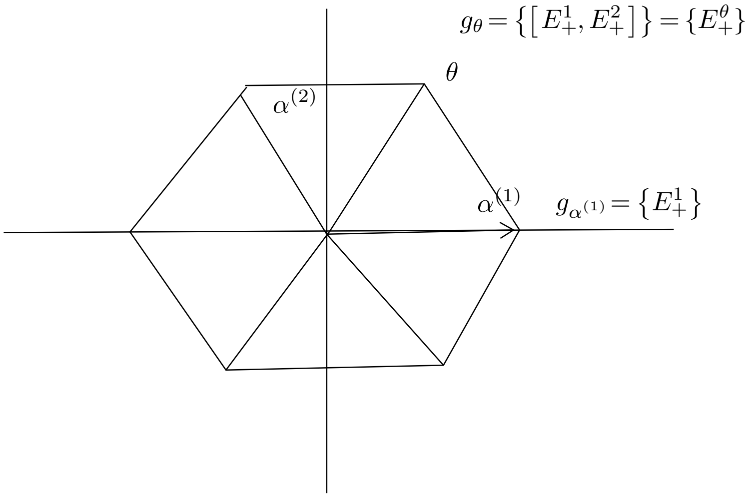
2009–12–02 Lecturer: Bengt E W Nilsson

Recall:



root diagram

$$\left[H^i, H^j\right] = 0, \quad i = 1, 2 \text{ in the Cartan subalgebra}$$

$$\left[H^i, E_\pm^j\right] = \left(\alpha^{(j)}\right)^i E_\pm^j$$

Root lattice $= \{n_1\,\alpha^{(1)} + n_2\,\alpha^{(2)}\}, n_i \in \mathbb{Z}$.

$$\alpha^{(i)} \cdot \alpha^{(j)} = A^{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Weight lattice = dual lattice of the roots = $\{m^1\,\Lambda_{(1)} + m^2\,\Lambda_{(2)}\}, m_i \in \mathbb{Z}$ and $\Lambda_{(i)} \cdot \alpha^{(j)} = \delta_i^j$.

$$\Lambda_{(i)} \cdot \Lambda_{(j)} = A_{ij}, \quad \text{where } A_{ij}A^{jk} = \delta_i^k \ \text{ (this is just notation!)}$$

Also

$$\alpha^{(i)} = A^{ij}\Lambda_{(j)}$$

$\Lambda_{(i)}$ are the fundamental weights.

$$\theta = \Lambda_{(1)} + \Lambda_{(2)} = \alpha^{(1)} + \alpha^{(2)}$$

Representation theory: Start from a highest weight state v_Λ (or vector in the module, which we normally just call representation, because we are sloppy).

$$H^i v_\Lambda = \Lambda^i\, v_\Lambda$$

$$\Lambda^i = \text{linear combination of } \Lambda_{(i)}$$

$\theta = \Lambda_{(1)} + \Lambda_{(2)}$ is the heighest weight.

Heighest weight state means that $E_+^i v_\Lambda = 0$.

State space:

$$\begin{array}{rll} & E_-^i v_\Lambda & E_-^i E_-^j v_\Lambda \\ \text{weights:} & \Lambda - \alpha^{(i)} & \Lambda - \alpha^{(i)} - \alpha^{(j)} \end{array}$$

Dynkin labels = (p, q) means that the heighest weight is $p\,\Lambda_{(1)} + q\,\Lambda_{(2)}$ where p, q are non-negative integers.

Read sections 11.1–11.3 in the book.

Unitarity of $\mathfrak{sl}(3,\mathbb{R})$ *and* $\mathfrak{su}(3)$.

$$\text{Note: } \mathfrak{sl}(3,\mathbb{R}) \overset{\text{complexify}}{\longrightarrow} \mathfrak{sl}(3,\mathbb{C}) \begin{array}{lll} \to & \mathfrak{sl}(3,\mathbb{R}) & \text{non-compact (split)} \\ \to & \mathfrak{su}(3) & \text{compact} \\ \to & \mathfrak{su}(2,1) & \text{non-compact} \end{array}.$$

Recall the generators of $\mathfrak{sl}(3,\mathbb{C}) \equiv A_2$ (Cartan classification).

$$H^1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad H^2 = \ldots$$

$$E_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ & & \\ & & \end{pmatrix}, \quad E_-^1 = \begin{pmatrix} 0 & & \\ 1 & & \\ 0 & & \end{pmatrix}, \quad \text{etc}$$

1) For $\mathfrak{su}(3)$ we need hermitian combinations H^i (ok), $E_+^i + E_-^i$ (hermitian), $\mathrm{i}(E_+^i - E_-^i)$ (hermitian). $\Rightarrow$ All unitary representations of $\mathfrak{su}(3)$.

$\mathfrak{su}(3)$: all representations are finite dimensional and unitary.

2) $\mathfrak{sl}(3, \mathbb{R})$. Finite dimensional representations: never unitary. Infinite dimensional representations: all known, and come in two classes: unitary ones, and non-unitary ones. The unitary ones are highest weight representations, but no lowest weight state exists. The non-unitary ones can be highest-weight representations, and non-highest-weight representations.

Levi's theorem about Lie algebras

A general Lie algebra can be decomposed as

$$\mathfrak{g} = \underbrace{V_-^{\text{compact}} \oplus V_+^{\text{non-compact}}}_{\text{semi-simple part}} \underbrace{\oplus}_{\substack{\text{direct}\\\text{or}\\\text{semi-}\\\text{direct}\\\text{sum}}} \underbrace{V_0^{\text{abelian}} \oplus V_0^{\text{nilpotent}}}_{\substack{\text{maximal solvable}\\\text{(the radical)}}} =$$

$$= \underbrace{\mathfrak{g}_1}_{\text{simple}} \oplus \underbrace{\mathfrak{g}_2}_{\text{simple}} \oplus \cdots \quad \begin{pmatrix} a & \alpha & \beta & \gamma \\ & b & \delta & \ldots \\ & & c & \ldots \\ & & & d \end{pmatrix}$$

Example: Poincaré algebra = $\mathrm{SO}(3) \oplus \{\text{translation}\}$. $[\Lambda^{\mu\nu}, p^\rho] = \delta^{\rho[\mu} p^{\nu]}$.

Exercise: Analyse the Lie algebra

$$\begin{pmatrix} 0 & a & \hbar \\ 0 & N & a^\dagger \\ 0 & 0 & 0 \end{pmatrix} = a\,X_a + a^\dagger X_{a^\dagger} + \hbar X_\hbar + N\,X_N$$

Compte the Killing form!

1) $\mathcal{H}=\left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & & \\ 0 & & 0 \end{array}\right)$. Lower right block: remove the corresponding operators.

2) Compute $\mathcal{H}$ again. $\mathcal{H}=0$.

Chapter 6: General theory of Lie algebras and their representations.

(Read Fuchs and Schweigert chapters 4 and 5 — a number of definitions. Check direct sum $\oplus$, Kronecker product $\times$, tensor product $\otimes$, section 5.6.)

Direct sum: $A_M=(A_\mu, A_m)$. A_M has $d=10$. A_μ has $d=4$. A_m has $d=6$. $10=4+6$. For tensor product we have $d=d_1\times d_2$.

SO(N) vector representation $\left(\Lambda^{MN}\right)_{PQ}=\delta^{MN}_{PQ}$.

The goal of chapter 6 in Fuchs and Schweigert is to show that any simple Lie algebra can be written in the form we wrote $\mathfrak{sl}(3,\mathbb{R})$.

1) $[H^i, H^j]=0, H^i\in\mathfrak{g}_0$ (Cartan sub-algebra).

2) $[H^i, E^j_\pm]=\pm A^{ji}E^j_\pm \;\Rightarrow$ root vectors as $(\alpha^{(j)})^i=A^{ji}$.

3) $\alpha^{(i)}\cdot\alpha^{(j)}=A^{ij}$

4) $\mathrm{tr}(H^iH^j)=A^{ij}$

5) In fact also:

$$\underbrace{\left[E^1_+,\left[E^1_+,E^2_+\right]\right]}_{\text{root string}}=0 \quad\Leftrightarrow\quad \left(\mathrm{ad}_{E^1_+}\right)^{\overbrace{1-A^{21}}^{=2}} E^2_+=0$$

$$\mathrm{ad}_x(y)=[x,y]$$

The Cartan matrix A^{ij} determines the complete algebra!

Fuchs and Schweigert § 6.1: Cartan subalgebra

- maximal set of mutually commuting generators (Cartan subalgebra) $\mathfrak{g}_0\equiv\mathrm{span}_\mathbb{C}\{H^i|i=1,\ldots,r\}$ where r is the rank. The span is over $\mathbb{C}$: take any $x\in\mathfrak{g}_0$, $T^a\in\mathfrak{g}$ (not $\mathfrak{g}_0$):

$$[x,T^a]=f^{xa}{}_b\,T^b$$

Diagonalisable.

$$[x,T^a]=f^xT^a$$

Characteristic equation:

$$\det((f^x)^a{}_b-f^x\,\delta^a{}_b)=0$$

To be able to solve this in any case we must use an algebraically complete number field, like $\mathbb{C}$ (not $\mathbb{R}$).

But note: There is always the split case where the $\mathfrak{sl}(2,\,\mathbb{R})$ subalgebras $\Rightarrow$ roots are real and even integers.

Remark: Characteristic equation $\rightarrow$ Cayley–Hamilton:

$$M^n+\lambda_1M^{n-1}+\lambda_2\,M^{n-2}+\cdots+\det M=0$$

λ_i's are invariants, such as $\operatorname{tr} M, \operatorname{tr} M^2, (\operatorname{tr} M)^2, \ldots$

Number of invariants is equal to the rank. These invariants are like Cassimir operators.

SU(2) L^i. $C_2 = L^i L^i \to j(j-1)$.

SU(3) $C_2 \,\&\, C_3 \to (p,q)$.

Fuchs and Schweigert § 6.2: roots.

$$[H^i, E^j] = \left(\alpha^{(j)}\right)^i E^j \quad \xrightarrow{\text{new notation}} \quad [H^i, E^\alpha] = \alpha^i E^\alpha$$

$$h = h_i H^i \Rightarrow [h, y] = \alpha_y(h) y$$

roots $\in \mathfrak{g}_0^*$ (dual df $\mathfrak{g}_0$).

Recall:

$$\mathcal{H}^{ij} \to h_i \mathcal{H}^{ij} h_j = h \cdot h = h_i\, h_j \operatorname{tr}(H^i H^j)$$

$$\alpha^{(i)} \cdot \alpha^{(j)} = (\alpha^{(i)})^m (\alpha^{(j)})^n \mathcal{H}_{mn}$$

Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \neq 0} g_\alpha \right)$$

Root system (diagram) $=$ $\{\text{all roots}\} = \Phi(g)$

§ 6.3 Killing form.

$\mathfrak{sl}(2, \mathbb{R})$

1) two-dimensional representation

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T^a = \{H, E_+, E_-\}$$

$$\to \operatorname{Tr}(T^a T^b) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2) three-dimensional representation

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_- = (E_+)^T$$

$$\operatorname{Tr}(T^a T^b) = 4 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition of the Killing form in terms of the algebra, i.e. the structure constants:

$$\mathcal{H}(T^a, T^b) \equiv \operatorname{tr}(\operatorname{ad}_{T^a} \circ \operatorname{ad}_{T^b}) \quad \text{where } \operatorname{ad}_{T^a}(T^b) = [T^a, T^b] = f^{ab}{}_c\, T^c$$

So $\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}(T^c) = \mathrm{ad}_{T^a}\big(f^{bc}{}_d T^d\big) = f^{bc}{}_d\, f^{ab}{}_e\, T^e$. Definition:

$$\mathrm{tr}\,(\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b}) = \mathrm{tr}_{c=e}\big(f^{bc}{}_d\, f^{ad}{}_e\big)$$

$(T^a)^b{}_c = -\, f^{ab}{}_c$ adjoint representation.

$$\mathcal{H}(T^a, T^b) = I_{\mathrm{ad}} \mathcal{H}^{ab}$$

I_{ad} depends on the representation.

So

$$\mathcal{H}^{ab} \equiv \frac{\mathcal{H}(T^a, T^b)}{I_{\mathrm{ad}}} = \frac{\mathcal{H}(T^a, T^b)}{I_{\mathrm{repr}}} \quad (I_{\substack{\mathrm{small} \\ \mathrm{rep}}} = 1)$$

$\mathcal{H}(T^a, T^b)$: any representation.

Note: $\mathcal{H}^{ab}$ is non-degenerate ($\det \mathcal{H}^{ab} \neq 0$) only if $\mathfrak{g}$ is semi-simple.

Question: How can we tell if $\mathfrak{g}$ is simple or semi-simple? That requires Dynkin diagrams.

Note: $\mathcal{H}^{ab}$ is invariant under the Lie algebra itself.

Note: $\mathcal{H}^{ab}$ is block diagonal in $\mathfrak{g}_0$ and the rest

$$\left(\begin{array}{c|c} \mathcal{H}^{ij} & 0 \\ \hline 0 & \begin{array}{c}\text{the} \\ \text{rest}\end{array} \end{array}\right)$$

§ 6.4. Properties of roots and root systems.

Nothing new.

§ 6.5. Structure constants in the Cartan–Weyl basis (refers to all generators and their commutators).

$$\big[H^i, H^j\big] = 0, \quad i = 1, 2, \ldots, \mathrm{rank}$$

$$\big[H^i, E^\alpha\big] = \alpha^i E^\alpha, \quad \alpha \in \Phi$$

$$\Rightarrow \quad \Big[H^1, \Big[E_+^\alpha, E_+^\beta\Big]\Big] = \big(\alpha^i + \beta^i\big)[\,\cdot\,,\cdot\,]$$

1) $\alpha^i + b^i \neq 0$ i.e. $\alpha + \beta \in \Phi$.

$$\big[E^\alpha, E^\beta\big] = e_{\alpha,\beta} E^{\alpha+\beta}$$

2) $\alpha + \beta = 0$ (origin) $\Rightarrow$ Cartan subalgebra. $\big[E^\alpha, E^{-\alpha}\big] \sim H \equiv \sum_{i=1}^{\mathrm{rank}} \hat{\alpha}_i H^i$

3) $\alpha + \beta \notin \Phi \Rightarrow \big[E^\alpha, E^\beta\big] = 0$.

Fuchs and Schweigert 6.6 and 6.7

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$$

raising operator (upper triangular), Cartan subalgebra (diagonal), lowering (lower triangular).

Definition: Simple roots.

EXAMPLE: A_2: $\alpha^{(1)}, \alpha^{(2)}$ simple. θ not simple.

Simple roots are chosen such that

— They are positive roots

— All other positive roots are linear combinations with non-negative integral coefficients.

Implication: $\alpha^{(i)} - \alpha^{(j)}$ is never a root if $\alpha^{(i)}$ and $\alpha^{(j)}$ are simple.

The rest of chapter 6 in Fuchs and Schweigert

To get the Chevally basis we use only the Cartan subalgebra and the simple roots, i.e. prove that any simple Lie algebra can be described entirely by just teh Cartan matrix A^{ij}.

Start.

$$[H^i, H^j] = 0, \quad i, j = 1, \ldots, \text{rank}$$

$$\left[H^i, E_\pm^j\right] = \pm \left(\alpha^{(j)}\right)^i E_\pm^j,$$

$$\text{tr}\left(H^i H^j\right) = \varkappa^{ij}$$

$$\alpha^{(i)} \cdot \alpha^{(j)} = G^{ij} \quad (\text{with}\, \varkappa^{-1} \text{ as metric})$$

Since we know that for some algebras (ex G_2, B_2) the roots have different length. $\Rightarrow$ It will be useful to normalise all roots to $(\text{length})^2 = 2$. Introduce coroots $\check{\alpha}^{(1)} \equiv \frac{\alpha^{(i)}}{c^{(i)}}$, where the $c^{(i)}$ are just numbers. Then we use the coroots to define the weight lattice.

$$\underbrace{\check{\alpha}^{(i)} \cdot \check{\alpha}^{(j)}}_{\text{with } \varkappa^{-1}} = \check{G}^{ij}$$

$$\Lambda_{(i)} \cdot \check{\alpha}^{(j)} = \delta_i^j$$

$$\underbrace{\Lambda_{(i)} \cdot \Lambda_{(j)}}_{\text{with } \varkappa} = \check{G}_{ij}$$

In fact we "know" from $\mathfrak{sl}(3, \mathbb{R})$ that if we set $\varkappa = \check{G}$ and $\check{\alpha} = \text{rows of } \check{G}$, then

$$\check{\alpha}^{(i)} \cdot \check{\alpha}^{(j)} = \check{G}^{ij}$$

$$\left(\check{G}\right)^{(i)n} \left(\check{G}\right)_{nm} \left(\check{G}^{(j)}\right)^m \equiv \check{G}^{(i)(j)}$$

That this is possible always follows from

$$H^{(i)} \equiv H^j \varkappa_{jk} \left(\check{\alpha}^{(i)}\right)^k$$

$$\text{tr}\, H^{(i)} H^{(j)} = \underbrace{\left(\text{tr}\, H^k H^l\right)}_{\varkappa^{kl}} \varkappa_{km} \left(\check{\alpha}^{(i)}\right)^m \varkappa_{ln} \left(\check{\alpha}^{(j)}\right)^n = \check{G}^{ij}$$

Step 2:

$$\left[H^{(i)}, E_\pm^j\right] = \pm \left(\alpha^{(j)}\right)^{(i)} E_\pm^j \equiv \left(\alpha^{(j)}\right)^m \kappa_{mn} \left(\check{\alpha}^{(i)}\right)^m E_\pm^j = \underbrace{\alpha^{(j)} \cdot \check{\alpha}^{(i)}}_{\text{with } \varkappa} E_\pm^j$$

Note: in Fuchs and Schweigert:

$$\alpha^{(i)} \cdot \breve{\alpha}^{(j)} \equiv \left(\alpha^{(i)}, \breve{\alpha}^{(j)}\right)$$

Step 3: Define the Cartan matrix

$$A^{ij} \equiv \alpha^{(i)} \cdot \breve{\alpha}^{(j)}$$

(not symmetric in general). It is symmetrisable $\rightarrow \ldots$

Step 4: We want all roots to have "(length)$^2 = 2$" to relate them to $\mathfrak{sl}(2, \mathbb{R})$ algebras.

$$\Rightarrow \breve{\alpha}^{(i)} = \frac{2\,\alpha^{(i)}}{\alpha^{(i)} \cdot \alpha^{(i)}}$$

Step 5:

$$\left[E_+^i, E_-^j\right] =$$

This has root $= \alpha^{(i)} - \alpha^{(j)}$, and this is never a root if $i \neq j$.

$$\left[E_+^i, E_-^j\right] = \delta^{ij} H^i$$

I choose coefficent $= 1$ by renormalising the $E_\pm^i$.

Chevally basis (page $\xi 8$ in FS)

$$\left[H^i, H^j\right] = 0$$

$$\left[H^i, E_\pm^j\right] = \pm\, A^{ji} E_\pm^i$$

$$\left[E_+^i, E_-^j\right] = \delta^{ij} H^i$$

Serre relations:

$$\left(\mathrm{ad}_{E_\pm^i}\right)^{1 - A^{ji}} E_\pm^j = 0$$